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MATH 172 Exam 1 Spring 2023

Sections 502 Solutions P. Yasskin

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|---|---|---|---|
| 1-7 | /49 | 9 | /20 |
| 8 | /15 | 10 | /20 |
| | | Total | /104 |

Multiple Choice: (7 points each. No part credit. Circle your answers.)

**1**. Find the area between the curves $x = y^2$ and $x = 2y$.

**a**. $\frac{2}{3}$

**b**. $\frac{4}{3}$ Correct

**c**. $\frac{8}{3}$

**d**. $\frac{16}{3}$

**e**. $\frac{32}{3}$

**Solution**: The curves intersect when $y^2 = 2y$ or $y = 0,2$. At $y = 1$, we have $y^2 = 1$ and $2y = 2$. So $2y$ is bigger and the area is

$$A = \int_0^2 (2y - y^2)\,dy = \left[ y^2 - \frac{y^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

**2**. Compute the average value of $f(x) = e^x$ on the interval $[0,2]$.

**a**. $\frac{1}{2}(e^2 + 1)$

**b**. $\frac{1}{4}(e^2 + 1)$

**c**. $\frac{1}{4}e^2$

**d**. $\frac{1}{2}(e^2 - 1)$ Correct

**e**. $\frac{1}{2}e^2$

**Solution**: $f_{ave} = \frac{1}{2}\int_0^2 e^x\,dx = \frac{1}{2}[e^x]_0^2 = \frac{1}{2}(e^2 - 1)$

**3**. Compute the right Riemann sum $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+i\frac{2}{n}\right)^3\frac{2}{n}$ by converting it into an integral and using the FTC to compute the integral.

**a**. $20$

**b**. $40$

**c**. $60$ Correct

**d**. $320$

**e**. $960$

**Solution**: A right Riemann sum has the form $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$.

Comparing the given sum to the general sum, we have $f(x_i) = \left(2+i\frac{2}{n}\right)^3$ and $\Delta x = \frac{2}{n}$.

So $f(x) = x^3$ and $x_i = 2 + i\frac{2}{n}$ and $b - a = 2$. So $a = 2$ and $b = 4$.

So the Riemann sum is the integral

$$\int_2^4 x^3\,dx = \left[\frac{x^4}{4}\right]_2^4 = \frac{4^4}{4} - \frac{2^4}{4} = 64 - 4 = 60$$

**4**. Compute $\int x\cos(4x)\,dx$.

**a**. $\frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x) + C$ Correct

**b**. $\frac{x}{4}\sin(4x) - \frac{1}{4}\cos(4x) + C$

**c**. $\frac{x}{4}\sin(4x) - \frac{1}{16}\cos(4x) + C$

**d**. $4x\sin(4x) + 16\cos(4x) + C$

**e**. $4x\sin(4x) - 4\cos(4x) + C$

**Solution**: Integrate by parts with

$$u = x \qquad dv = \cos(4x)dx$$
$$du = dx \qquad v = \frac{\sin(4x)}{4}$$

$$\int x\cos(4x)\,dx = \frac{x}{4}\sin(4x) - \frac{1}{4}\int\sin(4x)\,dx = \frac{x}{4}\sin(4x) - \frac{1}{4}\left(-\frac{\cos(4x)}{4}\right) + C$$
$$= \frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x) + C$$

**5**. Compute $\int e^x \sin(3x)\,dx$.

**a**. $-\frac{1}{8}e^x \sin(3x) + \frac{3}{8}e^x \cos(3x) + C$

**b**. $-\frac{1}{8}e^x \cos(3x) - \frac{3}{8}e^x \sin(3x) + C$

**c**. $e^x \sin(3x) - 3e^x \cos(3x) + C$

**d**. $\frac{1}{10}e^x \sin(3x) - \frac{3}{10}e^x \cos(3x) + C$ Correct

**e**. $\frac{1}{10}e^x \cos(3x) - \frac{3}{10}e^x \sin(3x) + C$

**Solution**: Integrate by parts with $\begin{array}{ll} u = \sin(3x) & dv = e^x dx \\ du = 3\cos(3x)dx & v = e^x \end{array}$

$$I = \int e^x \sin(3x)\,dx = e^x \sin(3x) - 3\int e^x \cos(3x)\,dx$$

Now integrate by parts with $\begin{array}{ll} u = \cos(3x) & dv = e^x dx \\ du = -3\sin(3x)dx & v = e^x \end{array}$

$$I = \int e^x \sin(3x)\,dx = e^x \sin(3x) - 3\left[e^x \cos(3x) + 3\int e^x \sin(3x)\,dx\right] = e^x \sin(3x) - 3e^x \cos(3x) - 9I$$

$$10I = e^x \sin(3x) - 3e^x \cos(3x) \qquad I = \frac{1}{10}e^x \sin(3x) - \frac{3}{10}e^x \cos(3x) + C$$

**6**. Compute $\int \tan^7\theta \sec^2\theta\,d\theta$.

**a**. $\frac{1}{16}\tan^8\theta \sec^2\theta + C$

**b**. $\frac{1}{24}\tan^8\theta \sec^3\theta + C$

**c**. $-\frac{\tan^8\theta}{2} + C$

**d**. $\frac{\tan^8\theta}{2} + C$

**e**. $\frac{\tan^8\theta}{8} + C$ Correct

**Solution**: We make the substitution $u = \tan\theta$ and $du = \sec^2\theta\,d\theta$. Then

$$\int \tan^7\theta \sec^2\theta\,d\theta = \int u^7\,du = \frac{u^8}{8} + C = \frac{\tan^8\theta}{8} + C$$

**7**. $\int_0^{\pi} \sin^2(3\theta)\cos^2(3\theta)\,d\theta$

**a**. $2\pi$

**b**. $\frac{1}{8}\pi$ Correct

**c**. $2\pi - \frac{1}{6}$

**d**. $2\pi + \frac{1}{6}$

**e**. $\frac{1}{8}\pi - \frac{1}{96}$

**Solution**: We use the identity $\sin(2A) = 2\sin A\cos A$ with $A = 3\theta$.

So $\sin(3\theta)\cos(3\theta) = \frac{1}{2}\sin(6\theta)$.

$I = \int_0^{\pi} \sin^2(3\theta)\cos^2(3\theta)\,d\theta = \frac{1}{4}\int_0^{\pi} \sin^2(6\theta)\,d\theta$

Now we use the identity $\sin^2 A = \dfrac{1-\cos(2A)}{2}$ with $A = 6\theta$.

So $\sin^2(6\theta) = \dfrac{1-\cos(12\theta)}{2}$.

$I = \frac{1}{8}\int_0^{\pi}(1-\cos(12\theta))\,d\theta = \frac{1}{8}\left[\theta - \frac{\sin(12\theta)}{12}\right]_0^{\pi} = \frac{1}{8}\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

**8**. (15 points) A bar of length $1$ m has linear density $\delta = \dfrac{1}{(1+x)^3}$ kg/m where $x$ is measured from one end.

**a**. Find the total mass of the bar.

**Solution**: $M = \int \delta\, dx = \int_0^1 \frac{1}{(1+x)^3}\, dx = \left[\frac{-1}{2(1+x)^2}\right]_0^1 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$

If necessary, use the substitution $u = 1 + x$.

**b**. Find the center of mass of the bar.

**Solution**: $M_1 = \int x\delta\, dx = \int_0^1 \frac{x}{(1+x)^3}\, dx$ Use the substitution $u = 1 + x$ and $du = dx$ and $x = u - 1$.

$$M_1 = \int_1^2 \frac{u-1}{u^3}\, du = \int_1^2 \frac{1}{u^2} - \frac{1}{u^3}\, du = \left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}$$

$$\bar{x} = \frac{M_1}{M} = \frac{1}{8}\frac{8}{3} = \frac{1}{3}$$

**9**. (20 points) A balloon is descending straight down with acceleration $a(t) = -20e^{-t}\ \dfrac{\text{km}}{\text{min}^2}$ where $t$ is in minutes.

It has initial height $y(0) = 2000$ km and initial velocity $v(0) = -10\ \dfrac{\text{km}}{\text{min}}$.

**a**. Find its velocity at $t = 2$ min.

**Solution**:

$$\frac{dv}{dt} = a(t) = -20e^{-t} \qquad v(t) = 20e^{-t} + C \qquad v(0) = 20 + C = -10 \qquad C = -30$$

$$v(t) = 20e^{-t} - 30 \qquad v(2) = 20e^{-2} - 30 = -30 + \frac{20}{e^2}$$

**b**. Find its height at $t = 2$ min.

**Solution**:

$$\frac{dy}{dt} = v(t) = 20e^{-t} - 30 \qquad y(t) = -20e^{-t} - 30t + K \qquad y(0) = -20 + K = 2000 \qquad K = 2020$$

$$y(t) = -20e^{-t} - 30t + 2020 \qquad y(2) = -20e^{-2} - 60 + 2020 = 1960 - \frac{20}{e^2}$$

**10**. (20 points) Consider the curve $\vec{r}(t) = \left( \frac{4}{3} t^{3/2}, t - \frac{t^2}{2} \right)$ between $t = 0$ and $t = 1$.

**a**. Find the arclength.

**Solution**: $\frac{dx}{dt} = \frac{4}{3}\frac{3}{2} t^{1/2} = 2t^{1/2} \qquad \frac{dy}{dt} = 1 - t$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(2t^{1/2})^2 + (1-t)^2}\,dt = \sqrt{4t + (1 - 2t + t^2)}\,dt = \sqrt{1 + 2t + t^2}\,dt$$

$$= \sqrt{(1+t)^2}\,dt = (1+t)\,dt$$

$$L = \int ds = \int_0^1 (1+t)\,dt = \left[ t + \frac{t^2}{2} \right]_0^1 = 1 + \frac{1}{2} = \frac{3}{2}$$

**b**. If the curve is rotated about the $x$-axis, find the surface area swept out.

**Solution**: $r = y = t - \frac{t^2}{2} \qquad ds = (1+t)\,dt$

$$A = \int 2\pi r\,ds = \int_0^1 2\pi\left(t - \frac{t^2}{2}\right)(1+t)\,dt = \int_0^1 \pi(2t - t^2)(1+t)\,dt = \int_0^1 \pi(2t + 2t^2 - t^2 - t^3)\,dt\backslash$$

$$= \int_0^1 \pi(2t + t^2 - t^3)\,dt = \pi\left[ t^2 + \frac{t^3}{3} - \frac{t^4}{4} \right]_0^1 = \pi\left(1 + \frac{1}{3} - \frac{1}{4}\right) = \frac{13}{12}\pi$$