Name________________________________

MATH 172 Exam 3 Spring 2023

Sections 502 Solutions P. Yasskin

1-7	/35	13	/20
8-12	/50	Total	/105

Multiple Choice: (5 points each. No part credit. Circle your answers.)

1. $L = \lim\limits_{n\to\infty} \dfrac{\dfrac{4}{n^2} - \dfrac{9}{n^3}}{\dfrac{2}{n^2} + \dfrac{3}{n^3}} =$

a. -3

b. 0

c. 2 Correct

d. 3

e. diverges

Solution: For large n, $\dfrac{1}{n^2} > \dfrac{1}{n^3}$. So:

$$\lim_{n\to\infty} \frac{\frac{4}{n^2} - \frac{9}{n^3}}{\frac{2}{n^2} + \frac{3}{n^3}} = \lim_{n\to\infty} \frac{\frac{4}{n^2} - \frac{9}{n^3}}{\frac{2}{n^2} + \frac{3}{n^3}} \frac{n^2}{n^2} = \lim_{n\to\infty} \frac{4 - \frac{9}{n}}{2 + \frac{3}{n}} = \frac{4-0}{2+0} = 2$$

2. $L = \lim\limits_{n\to\infty} \left(\dfrac{1}{n^2}\right)^{\frac{2}{\ln n}} =$

a. -4

b. $\ln 4$

c. $-\ln 4$

d. e^{-4} Correct

e. diverges

Solution: $\ln L = \lim\limits_{n\to\infty} \ln\left(\dfrac{1}{n^2}\right)^{\frac{2}{\ln n}} = \lim\limits_{n\to\infty} \dfrac{2}{\ln n} \ln\left(\dfrac{1}{n^2}\right) = \lim\limits_{n\to\infty} \dfrac{-4\ln n}{\ln n} = -4$ So $L = e^{-4}$

3. $L = \lim\limits_{n\to\infty} n^2\left[1 - \cos\left(\frac{1}{n}\right)\right] =$

a. $\dfrac{1}{2}$ Correct

b. 2

c. -2

d. $-\dfrac{1}{2}$

e. diverges

Solution: Let $t = \frac{1}{n}$. Then use l'Hopital's Rule:

$$\lim_{n\to\infty} n^2\left[1 - \cos\left(\frac{1}{n}\right)\right] = \lim_{t\to 0^+} \frac{1-\cos t}{t^2} \overset{l'H}{=} \lim_{t\to 0^+} \frac{\sin t}{2t} \overset{l'H}{=} \lim_{t\to 0^+} \frac{\cos t}{2} = \frac{1}{2}$$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{3^n} =$

- **a.** $\frac{4}{15}$ Correct
- **b.** $\frac{3}{5}$
- **c.** $\frac{4}{3}$
- **d.** $-\frac{2}{5}$
- **e.** divergent

Solution: $a = \frac{2^2}{3^2} = \frac{4}{9} \qquad r = \frac{-2}{3} \qquad |r| = \frac{2}{3} < 1 \qquad \displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{3^n} = \frac{\frac{4}{9}}{1 - \frac{-2}{3}} = \frac{4}{9+6} = \frac{4}{15}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{2^{2n}}{3^n} =$

- **a.** $-\frac{16}{3}$
- **b.** $\frac{4}{7}$
- **c.** $\frac{16}{3}$
- **d.** $\frac{16}{21}$
- **e.** divergent Correct

Solution: $a = \frac{2^4}{3^2} = \frac{16}{9} \qquad r = \frac{2^2}{3} = \frac{4}{3} \qquad |r| = \frac{4}{3} > 1$ divergent

6. $\displaystyle\sum_{n=1}^{\infty} [\arctan(n) - \arctan(n+1)] =$

- **a.** $-\frac{\pi}{2}$
- **b.** $-\frac{\pi}{4}$ Correct
- **c.** 0
- **d.** $\frac{\pi}{4}$
- **e.** $\frac{\pi}{2}$

Solution: $S_k = \displaystyle\sum_{n=1}^{k} [\arctan(n) - \arctan(n+1)]$

$= [\arctan 1 - \arctan 2] + [\arctan 2 - \arctan 3] + \cdots + [\arctan k - \arctan(k+1)] = \arctan 1 - \arctan(k+1)$

$S = \lim\limits_{n\to\infty} S_k = \lim\limits_{n\to\infty} [\arctan 1 - \arctan(k+1)] = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

7. If $S = \sum_{n=2}^{\infty} a_n$ and $S_k = \sum_{n=2}^{k} a_n = \frac{k-1}{k}$, what is a_k?

HINT: What are S_{k-1} and $S_k - S_{k-1}$?

a. $a_k = \frac{k}{k-1}$

b. $a_k = \frac{k-1}{k}$

c. $a_k = \frac{k}{k+1}$

d. $a_k = \frac{1}{k(k+1)}$

e. $a_k = \frac{1}{k(k-1)}$ Correct

Solution: Since $S_k = \frac{k-1}{k}$, we compute $S_{k-1} = \frac{k-2}{k-1}$.

Then on the one hand, $S_k - S_{k-1} = \sum_{n=2}^{k} a_n - \sum_{n=2}^{k-1} a_n = a_k$

and on the other hand, $S_k - S_{k-1} = \frac{k-1}{k} - \frac{k-2}{k-1} = \frac{(k^2-2k+1)-(k^2-2k)}{k(k-1)} = \frac{1}{k(k-1)}$

So $a_k = \frac{1}{k(k-1)}$.

Short Answer: (10 points each. No part credit. Circle your answers.)

For each series, circle **Convergent** or **Divergent** and circle **the test you used**.
(Most series have more than one acceptable test. **Only circle one**!)
For a Comparison or Absolute Convergence Test, write the Comparison series or the Related Absolute series and the write the name of the test you used for that series.
For a Divergence or Integral or Limit Comparison or Ratio Test, compute the required limit or integral.
For a Simple Comparison Test, check the required inequality.

8. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^{3/2}-1}$ Convergent [✓ Divergent]

a. Geometric Series
b. Telescoping Series
c. n^{th} Term Divergence Test
d. [✓ Integral Test]
e. p-Series Test
f. [✓ Simple Comparison Test]
g. [✓ Limit Comparison Test]
h. Alternating Series Test
i. Absolute Convergence Test
j. Ratio Test

Comparison or Absolute Series________________ Its test:___________________________
Any required limit or integral or inequality:

Solution: For Integral: $\int_2^{\infty} \frac{\sqrt{n}}{n^{3/2}-1}\,dn = \left[\frac{2}{3}\ln(n^{3/2}-1)\right]_2^{\infty} = \infty$

Compare to : $\sum_{n=2}^{\infty} \frac{1}{n}$ harmonic p –series $L = \lim_{n\to\infty} \frac{\sqrt{n}}{n^{3/2}-1}\frac{n}{1} = 1 \quad 0 < L < \infty$

For Simple: $\frac{\sqrt{n}}{n^{3/2}-1} > \frac{\sqrt{n}}{n^{3/2}} = \frac{1}{n}$ For Limit: $L = \lim_{n\to\infty} \frac{\sqrt{n}}{n^{3/2}-1}\frac{n}{1} = 1 \quad 0 < L < \infty$

9. $\sum_{n=0}^{\infty} \frac{\arctan n}{1+n^2}$ [✓ Convergent] Divergent

a. Geometric Series

b. Telescoping Series

c. n^{th} Term Divergence Test

d. [✓ Integral Test]

e. p-Series Test

f. [✓ Simple Comparison Test]

g. [✓ Limit Comparison Test]

h. Alternating Series Test

i. Absolute Convergence Test

j. Ratio Test

Comparison or Absolute Series________________ Its test:____________________________

Any required limit or integral or inequality:

Solution: For Integral: $\int_0^{\infty} \frac{\arctan n}{1+n^2} = \left[\frac{1}{2}(\arctan n)^2\right]_0^{\infty} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2$

Or compare to $\sum_{n=0}^{\infty} \frac{\pi}{2}\frac{1}{n^2}$ p-series For Simple Comparison $\frac{\arctan n}{1+n^2} < \frac{\pi}{2}\frac{1}{n^2}$

For Limit Comparison $L = \lim_{n\to\infty} \frac{\arctan n}{1+n^2}\frac{2n^2}{\pi} = 1 \qquad 0 < L < \infty$

10. $\sum_{n=1}^{\infty} n!e^{-n}$ Convergent [✓ Divergent]

a. Geometric Series

b. Telescoping Series

c. [✓ n^{th} Term Divergence Test]

d. Integral Test

e. p-Series Test

f. Simple Comparison Test

g. Limit Comparison Test

h. Alternating Series Test

i. Absolute Convergence Test

j. [✓ Ratio Test]

Comparison or Absolute Series________________ Its test:____________________________

Any required limit or integral or inequality:

Solution: For Divergence: $\lim_{n\to\infty} n!e^{-n} = \lim_{n\to\infty} \frac{n}{e}\frac{(n-1)}{e}\cdots\frac{1}{e} = \infty$ since ∞ product of numbers > 1.

For Ratio: $\lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}}\frac{e^n}{n!} = \lim_{n\to\infty} \frac{(n+1)}{e} = \infty > 1$

11. $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$ [✓ Convergent] Divergent

- **a.** Geometric Series
- **b.** Telescoping Series
- **c.** n^{th} Term Divergence Test
- **d.** Integral Test
- **e.** p-Series Test
- **f.** Simple Comparison Test
- **g.** [✓ Limit Comparison Test]
- **h.** Alternating Series Test
- **i.** Absolute Convergence Test
- **j.** [✓ Ratio Test]

Comparison or Absolute Series________________ Its test:____________________________

Any required limit or integral or inequality:

Solution: For Ratio: $\lim\limits_{n\to\infty} \frac{2^{n+1}}{3^{n+1} - 1} \frac{3^n - 1}{2^n} = \frac{2}{3} < 1$

For Limit Comparison: Compare to $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ which is a convergent geometric series.

$$L = \lim_{n\to\infty} \frac{2^n}{3^n - 1} \frac{3^n}{2^n} = 1 \qquad 0 < L < \infty$$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^3}$ [✓ Convergent] Divergent

- **a.** Geometric Series
- **b.** Telescoping Series
- **c.** n^{th} Term Divergence Test
- **d.** Integral Test
- **e.** p-Series Test
- **f.** Simple Comparison Test
- **g.** Limit Comparison Test
- **h.** Alternating Series Test
- **i.** [✓ Absolute Convergence Test]
- **j.** Ratio Test

Comparison or Absolute Series________________ Its test:____________________________

Any required limit or integral or inequality:

Solution: absolute series is $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ which converges by Simple Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which is a p-series with $p = 3 > 1$ since $\frac{|\sin n|}{n^3} < \frac{1}{n^3}$.

Work Out: (Points indicated. Part credit possible. Show all work.)

13. $(20$ points$)$ Consider the recursively defined sequence $a_{n+1} = 5 - \dfrac{4}{a_n}$ with $a_1 = 2$

a. $(5$ pts$)$ Assuming the limit exists, find the possible values of the limit.

Solution: Assuming the limit exists, let $L = \lim\limits_{n\to\infty} a_n$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$ also.
Taking the limit of the recurrsion relation, we have

$$L = 5 - \frac{4}{L} \quad \Rightarrow \quad L^2 = 5L - 4 \quad \Rightarrow \quad L^2 - 5L + 4 = 0 \quad \Rightarrow \quad (L-1)(L-4) = 0$$

So if the limit exists, it must be $L = 1$ or $L = 4$.

b. $(2$ pts$)$ (Fill in the blanks.) Compute the first 3 terms of the sequence:

$a_1 =$ ___________ $a_2 =$ ___________ $a_3 =$ ___________

Solution: $a_1 = \underline{\quad 2 \quad}$ $a_2 = \underline{\quad 3 \quad}$ $a_3 = \underline{\dfrac{11}{3} \approx 3.67}$

c. $(2$ pt$)$ (Fill in the blanks.) The sequence appears to be bounded

below by __________ and above by __________.

Solution: below by $\underline{\quad 2 \quad \text{(or anything less)}}$, above by $\underline{\quad 4 \quad \text{(or anything more)}}$.

d. $(4$ pts$)$ Use induction to prove the sequence is bounded below and above by your answers.

Solution: We show $1 < a_n < 4$:
Initialization step: $1 < a_1 = 2 < 4$
Induction step: Assume $1 < a_k < 4$. Then

$$1 > \frac{1}{a_k} > \frac{1}{4} \quad \text{and} \quad -4 < -\frac{4}{a_k} < -1 \quad \text{and} \quad 5 - 4 < 5 - \frac{4}{a_k} < 5 - 1 \quad \text{or} \quad 1 < 5 - \frac{4}{a_k} < 4.$$

So $1 < a_{k+1} < 4$.

e. $(1$ pt$)$ (Circle one.) The sequence appears to be increasing decreasing.

Solution: $\boxed{\text{increasing}}$ decreasing.

f. $(4$ pts$)$ Use induction to prove the sequence is increasing or decreasing.

Solution: We show $a_n < a_{n+1}$:
Initialization step: $a_1 = 2 < a_2 = 3$
Induction step: Assume $a_k < a_{k+1}$. Then

$$\frac{1}{a_k} > \frac{1}{a_{k+1}} \quad \text{and} \quad -\frac{4}{a_k} < -\frac{4}{a_{k+1}} \quad \text{and} \quad 5 - \frac{4}{a_k} < 5 - \frac{4}{a_{k+1}}.$$

So $a_{k+1} < a_{k+2}$.

g. $(2$ pt$)$ What do you conclude about the limit of the sequence? Why?
What theorem are you using? (This must be sentences.)

Solution: By the Bounded Monotonic Sequence Theorem, the sequence has a limit.
Since the limit must be 1 or 4 and it increases from 2, the limit must be $L = 4$.

14. $\left(16 \text{ points}\right)$

Solution:

20 #22 div simp or lim comp w $\frac{1}{n}$ $\frac{\sqrt{n}}{n^{3/2}-1}$ *done*

20 #9 conv integ or simp or lim comp w 1/n^2 $\frac{\arctan n}{1+n^2}$ *done*

20 #34 div div or ratio $n!e^{-n}$ *done*

20 #18 conv ratio or lim comp w $\left(\frac{2}{3}\right)^n$ $\frac{2^n}{3^n-1}$

21 #13 conv rel abs $(-1)^n\frac{\sin n}{n^3}$ comp to $\frac{1}{n^3}$

21 #6 div div or ratio $\frac{(-1)^n 4^n}{4^{n+1}-1}$