Name________________________________

1-10	/50	13	/18
12	/18	14	/20
		Total	/106

MATH 172 Final Exam Spring 2023

Sections 502 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit. Circle your answers.)

1. $\int_0^\pi \sin^3 x\,dx =$

a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. $\frac{4}{3}$ Correct

d. $\frac{3}{8}\pi$

e. $\frac{3}{4}\pi$

Solution: Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$. So

$$\int_0^\pi \sin^3 x\,dx = \int_0^\pi (1-\cos^2 x)\sin x\,dx = -\int_1^{-1}(1-u^2)\,du = \left[-u + \frac{u^3}{3}\right]_1^{-1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

2. $\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx =$

a. $\frac{\sqrt{4x^2-9}}{9x} + C$ Correct

b. $\frac{9x}{\sqrt{4x^2-9}} + C$

c. $\frac{2\sqrt{4x^2-9}}{27} + C$

d. $\frac{4}{27}\ln\left(\frac{2x}{3} + \frac{\sqrt{4x^2-9}}{3}\right) + C$

e. $\frac{4}{27}\ln\frac{\sqrt{4x^2-9}}{2x} - \frac{1}{27}\frac{4x^2-9}{2x^2}$

Solution: Let $2x = 3\sec\theta$. Then $x = \frac{3}{2}\sec\theta$ and $dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$.

$$\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx = \int \frac{4}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}\frac{3}{2}\sec\theta\tan\theta\,d\theta == \int \frac{4}{9\sec^2\theta 3\tan\theta}\frac{3}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{2}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{2}{9}\int \cos\theta\,d\theta = \frac{2}{9}\sin\theta + C$$

Since $\sec\theta = \frac{2x}{3}$, we draw a triangle with hypotenuse $2x$ and adjacent side 3.

Then the opposite side is $\sqrt{4x^2-9}$ and $\sin\theta = \frac{\sqrt{4x^2-9}}{2x}$. So

$$\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx = \frac{\sqrt{4x^2-9}}{9x} + C$$

3. In the partial fraction expansion, $\frac{8}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$, which coefficient is **right**?

a. $A = 1$

b. $B = -2$ Correct

c. $B = 2$

d. $C = -2$

e. $C = 2$

Solution: Clear denominator: $8 = A(x^2+4) + (Bx+C)x$

$x = 0$: $8 = A(4)$ $\Rightarrow$ $A = 2$

$x = 1$: $8 = A(5) + B + C = 10 + B + C$ $\Rightarrow$ $B + C = -2$

$x = -1$: $8 = A(5) + B - C = 10 + B - C$ $\Rightarrow$ $B - C = -2$

Add: $2B = -4$ $\Rightarrow$ $B = -2$

Subtract: $2C = 0$ $\Rightarrow$ $C = 0$

4. Approximate $\int_2^{14} \frac{144}{x^2}\,dx$ using a midpoint Riemann sum with 3 intervals.

a. $\frac{49}{4}$

b. $\frac{74}{3}$

c. 62

d. 74

e. 49 Correct

Solution: The width of each interval is $\Delta x = \frac{14-2}{3} = 4$. The partition points are $2, 6, 10, 14$. The midpoints are $4, 8, 12$. With $f(x) = \frac{144}{x^2}$, the function values are $f(4) = \frac{144}{16} = 9$, $f(8) = \frac{144}{64} = \frac{9}{4}$, $f(12) = \frac{144}{144} = 1$. So the Riemann sum is

$$R_3 = \big(f(2) + f(8) + f(12)\big)\Delta x = \left(9 + \frac{9}{4} + 1\right)4 = 49$$

5. Find the arc length of the curve $(x,y,z) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $t = 0$ and $t = 1$.

a. $\frac{5}{3}$ Correct

b. $\frac{8}{3}$

c. $\frac{16}{3}$

d. 2

e. 4

Solution: The differential of arclength is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1^2 + (2t)^2 + (2t^2)^2}\,dt = \sqrt{1 + 4t^2 + 4t^4}\,dt$$

$$= \sqrt{(1+2t^2)^2}\,dt = (1+2t^2)\,dt$$

So the arclength is $L = \int_0^1 ds = \int_0^1 (1+2t^2)\,dt = \left[t + \frac{2t^3}{3}\right]_0^1 = 1 + \frac{2}{3} = \frac{5}{3}$

6. The curve $y = x^2$ between $x = 0$ and $x = \sqrt{2}$ is revolved about the y-axis. Find the area of the surface swept out.

a. 3π

b. $\frac{7}{4}\pi$

c. $\frac{9}{2}\pi$

d. 4π

e. $\frac{13}{3}\pi$ Correct

Solution: The surface area is $A = \int_0^{\sqrt{2}} 2\pi r\, ds$ where the radius is $r = x$ and the differential of arclength is $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (2x)^2}\, dx = \sqrt{1 + 4x^2}\, dx$. So $A = \int_0^{\sqrt{2}} 2\pi x \sqrt{1 + 4x^2}\, dx$. Let $u = 1 + 4x^2$. Then $du = 8x\,dx$ and $\frac{1}{8}\,du = x\,dx$. So $A = \frac{2\pi}{8}\int_1^9 \sqrt{u}\, du = \frac{\pi}{4}\left[\frac{2u^{3/2}}{3}\right]_1^9 = \frac{\pi}{6}(9^{3/2} - 1^{3/2}) = \frac{\pi}{6}(26) = \frac{13}{3}\pi$

7. A sequence is defined recursively by $a_1 = 3$ and $a_{n+1} = \frac{a_n{}^2 + 7}{8}$. Find $\lim_{n\to\infty} a_n$.

a. 0

b. 1 Correct

c. 2

d. 3

e. 7

Solution: Assuming the limit exists, let $L = \lim_{n\to\infty} a_n$. Then $\lim_{n\to\infty} a_{n+1} = L$ also. We solve

$$L = \frac{L^2 + 7}{8} \quad \Rightarrow \quad 8L = L^2 + 7 \quad \Rightarrow \quad 0 = L^2 - 8L + 7 = (L-1)(L-7) \quad \Rightarrow \quad L = 1, 7$$

The first few terms are $a_1 = 3$, $a_2 = \frac{9+7}{8} = 2$, $a_3 = \frac{4+7}{8} = \frac{11}{8}$.

So the sequence seems to be decreasing from 3. We could use induction to prove it is decreasing and bounded below by 0. So the limit must be $\lim_{n\to\infty} a_n = 1$.

8. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 + \sqrt{n}}{n^2 + \sqrt{n}}$ is:

a. Absolutely Convergent Correct

b. Conditionally Convergent

c. Divergent

d. Conditionally Divergent

Solution: The related absolute series is $\sum_{n=1}^{\infty} \frac{1 + \sqrt{n}}{n^2 + \sqrt{n}}$ which is convergent by comparison with $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2} > 1$. So the original series is absolutely convergent by the Absolute Convergence Test.

9. The series $\sum_{n=1}^{\infty} \frac{1+n}{n+n^4}$ is:

a. convergent by Simple Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

b. convergent by Limit Comparison but not Simple Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Correct

c. divergent by Simple Comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

d. divergent by Limit Comparison but not Simple Comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: For large n, we have $n > 1$ and $n^4 > n$. So we compare to $\sum_{n=1}^{\infty} \frac{n}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which is a p-series with $p = 3 > 1$, and so is convergent.

The Simple Comparison Test will not work because $1 + n > n$.

So we apply the Limit Comparison Test:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+n}{n+n^4}\frac{n^3}{1} = \lim_{n\to\infty} \frac{n^3+n^4}{n+n^4} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n^3}+1} = 1 \text{ and } 0 < L < \infty$$

10. $\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} =$

a. ∞

b. $\frac{1}{6}$

c. 0

d. $-\frac{1}{6}$ Correct

e. $-\infty$

Solution: We start with the Maclaurin series $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$. We substitute $u = x^3$:

$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$ and insert into the limit:

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9}$$

$$= \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^6}{5!} - \cdots\right) = -\frac{1}{6}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. $(18$ points$)$ The area below $y = e^{-x}$ between $x = 0$ and $x = 2$ is revolved about the y-axis. Find the volume of the solid swept out.

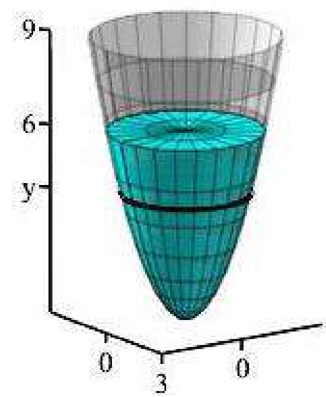
Solution: We do an x-integral. The slices are vertical and revolve into cylinders. The radius is $r = x$ and the height is $h = e^{-x}$. So the volume is $V = \int 2\pi rh\,dx = 2\pi \int_0^2 xe^{-x}\,dx$.

We integrate by parts with $\begin{matrix} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{matrix}$.

$$V = 2\pi \int_0^2 xe^{-x}\,dx = 2\pi\left[-xe^{-x} + \int e^{-x}\,dx\right]_0^2 = 2\pi\left[-xe^{-x} - e^{-x}\right]_0^2$$
$$= 2\pi(-2e^{-2} - e^{-2}) - 2\pi(-1) = 2\pi(1 - 3e^{-2})$$

12. $(18$ points$)$ The curve $y = x^2$ for $y \le 9$ is revolved about the y-axis to form a bowl. It is filled to a depth of $y = 6$ with salt water with weight density $g\delta = 64\ \dfrac{\text{lb}}{\text{ft}^3}$.

How much work is done to pump the water out the top of the bowl.



Solution: The slice at height y is lifted a distance $D = 9 - y$.
This slice is a disk of thickness dy and radius $r = x = \sqrt{y}$.
So its volume is $dV = \pi r^2\,dy = \pi y\,dy$. And its weight is $dF = g\delta dV = 64\pi y\,dy$.
So the work is $W = \int_0^6 D\,dF = \int_0^6 (9 - y)\,64\pi y\,dy = 64\pi\left[9\dfrac{y^2}{2} - \dfrac{y^3}{3}\right]_0^6 = 64\pi\left(9\dfrac{6^2}{2} - \dfrac{6^3}{3}\right) = 5760\pi$

13. $\left(20\right.$ points$\left.\right)$ Find the interval of convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{2^n + 4}{6^n + 12}(x-5)^n$ as follows:

a. Find the radius of convergence.

Solution: We use the ratio test: $|a_n| = \dfrac{(2^n+4)|x-5|^n}{(6^n+12)}$ $\quad |a_{n+1}| = \dfrac{(2^{n+1}+4)|x-5|^{n+1}}{(6^{n+1}+12)}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-5|^{n+1}}{|x-5|^n} \frac{(2^{n+1}+4)}{(2^n+4)} \frac{(6^n+12)}{(6^{n+1}+12)} = |x-5| \lim_{n\to\infty} \frac{\left(2+\frac{4}{2^n}\right)}{\left(1+\frac{4}{2^n}\right)} \lim_{n\to\infty} \frac{\left(1+\frac{12}{6^n}\right)}{\left(6+\frac{12}{6^n}\right)}$$

$= \frac{2}{6}|x-5| < 1 \quad \Rightarrow \quad |x-5| < 3 \quad \Rightarrow \quad R = 3 \quad \Rightarrow$ Open interval of convergence is: $(2,8)$

b. Check convergence at the right endpoint.

Solution: $x = 8$: $\quad \displaystyle\sum_{n=2}^{\infty} \frac{2^n+4}{6^n+12}(3)^n$ Diverges by the n^{th}Term Divergence Test because

$$\lim_{n\to\infty} \frac{2^n+4}{6^n+12}(3)^n = \lim_{n\to\infty} \frac{6^n + 4 \cdot 3^n}{6^n+12} = \lim_{n\to\infty} \frac{1+\frac{4}{2^n}}{1+\frac{12}{6^n}} = 1 \neq 0$$

c. Check convergence at the left endpoint.

Solution: $x = 2$: $\quad \displaystyle\sum_{n=2}^{\infty} \frac{2^n+4}{6^n+12}(-3)^n$ Diverges by the n^{th} Term Divergence Test because

$$\lim_{n\to\infty} \frac{2^n+4}{6^n+12}(-3)^n = \lim_{n\to\infty}(-1)^n \frac{6^n + 4 \cdot 3^n}{6^n+12} = \lim_{n\to\infty}(-1)^n \frac{1+\frac{4}{2^n}}{1+\frac{12}{6^n}} \neq 0 \quad \text{because}$$

the terms alternate between close to 1 and close to -1.

d. State the interval of convergrnce.

Solution: The interval of convergence is: $(2,8)$