Name____________________ Sec________

1-13	/65	15	/10
14	/21	16	/10
		Total	/106

MATH 251 Final Spring 2010

Sections 511 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the angle between the vectors $\vec{u} = (2,2,1)$ and $\vec{v} = (1,2,2)$.

a. $\arccos(8/9)$ Correct Choice

b. $\arccos(8/3)$

c. $\arccos\left(\sqrt{8}/9\right)$

d. $\arccos\left(3/\sqrt{8}\right)$

e. $\arccos(3/8)$

$\vec{u}\cdot\vec{v} = 2+4+2 = 8$ $\quad |\vec{u}| = \sqrt{4+4+1} = 3$ $\quad |\vec{v}| = \sqrt{1+4+4} = 3$ $\quad \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{8}{9}$

2. At the point (x,y,z) where the line $\vec{r}(t) = (1-t,t,2-2t)$ intersects the plane $x-2y+3z = 16$, we have $x+y+z =$

a. -2

b. 2

c. 3

d. 5 Correct Choice

e. 16

Plug the line into the plane and solve for t:

$(1-t)-2(t)+3(2-2t) = 16 \qquad -9t+7 = 16 \qquad t = -1$

Plug back into the line:

$(x,y,z) = (1-t,t,2-2t) = (2,-1,4)$ So $x+y+z = 5$

3. If a jet flies around the world from West to East, directly above the equator, in what direction does its unit binormal $\hat{B}$ point?

a. Down (toward the center of the earth)

b. Up (away from the center of the earth)

c. North Correct Choice

d. South

e. West

$\hat{T}$ points East, $\hat{N}$ points Down and by the right hand rule $\hat{B}$ points North.

4. Find the z-intercept of the plane tangent to the surface $\dfrac{xy}{z} = 1$ at the point $(2,3,6)$.

a. 6

b. $\dfrac{1}{6}$

c. 5

d. -5

e. -6 Correct Choice

$F = \dfrac{xy}{z} \qquad \vec{\nabla} F = \left(\dfrac{y}{z}, \dfrac{x}{z}, -\dfrac{xy}{z^2} \right) \qquad \vec{N} = \vec{\nabla} F \Big|_{(2,3,6)} = \left(\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{6} \right)$

$\vec{N} \bullet X = \vec{N} \bullet P \qquad \dfrac{1}{2}x + \dfrac{1}{3}y - \dfrac{1}{6}z = \dfrac{1}{2}2 + \dfrac{1}{3}3 - \dfrac{1}{6}6 = 1 \qquad \dfrac{1}{6}z = \dfrac{1}{2}x + \dfrac{1}{3}y - 1$

$z = 3x + 2y - 6$ $\quad$ z-intercept is $c = -6$.

5. The temperature in an ideal gas is given by $T = \kappa \dfrac{P}{\rho}$ where κ is a constant, P is the pressure and ρ is the density. At a certain point $Q = (3,2,1)$, we have

$$P(Q) = 8 \qquad \vec{\nabla} P(Q) = (4,-2,-4)$$
$$\rho(Q) = 2 \qquad \vec{\nabla} \rho(Q) = (-1,4,2)$$

So at the point Q, the temperature is $T(Q) = 4\kappa$ and its gradient is $\vec{\nabla} T(Q) =$

a. $\kappa(-8.5,6,9)$

b. $\kappa(4,-9,-6)$ Correct Choice

c. $\kappa(3,2,-2)$

d. $\kappa\left(\dfrac{1}{2},2\right)$

e. $\kappa\left(-\dfrac{1}{2},2\right)$

By chain rule: (Think about each component separately.)

$$\vec{\nabla} T = \frac{\partial T}{\partial P}\vec{\nabla} P + \frac{\partial T}{\partial \rho}\vec{\nabla} \rho = \frac{\kappa}{\rho}\vec{\nabla} P - \frac{\kappa P}{\rho^2}\vec{\nabla} \rho = \frac{\kappa}{2}(4,-2,-4) - \frac{\kappa 8}{2^2}(-1,4,2)$$
$$= \kappa(2,-1,-2) + \kappa(2,-8,-4) = \kappa(4,-9,-6)$$

6. If the temperature in a room is $T = xyz^2$, find the rate of change of the temperature as seen by a fly who is located at $(3, 2, 1)$ and has velocity $(1, 2, 3)$.

a. 32

b. 36

c. 44 Correct Choice

d. 48

e. 52

$$\frac{dT}{dt} = \vec{v} \cdot \vec{\nabla} T \Big|_{(3,2,1)} = (1,2,3) \cdot (yz^2, xz^2, 2xyz)|_{(3,2,1)} = (1,2,3) \cdot (2,3,12) = 44$$

7. Find the volume below $z = xy$ above the region between the curves $y = 3x$ and $y = x^2$.

a. $\frac{81}{2}$

b. $\frac{81}{4}$

c. $\frac{81}{8}$

d. $\frac{243}{2}$

e. $\frac{243}{8}$ Correct Choice

$3x = x^2 \quad \Rightarrow \quad x = 0, 3$

$$V = \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \left[\frac{xy^2}{2} \right]_{y=x^2}^{3x} dx = \int_0^3 \left(\frac{x9x^2}{2} - \frac{xx^4}{2} \right) dx = \left[\frac{9x^4}{8} - \frac{x^6}{12} \right]_{x=0}^{3} = \frac{3^6}{4} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{243}{8}$$

8. Compute $\iint_C e^{-x^2-y^2} dx\,dy$ over the disk enclosed in the circle $x^2 + y^2 = 4$.

a. $\frac{\pi}{2}(1 - e^{-4})$

b. $\pi(1 - e^{-4})$ Correct Choice

c. $\frac{\pi}{2} e^{-4}$

d. πe^{-4}

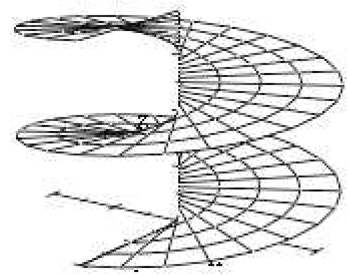
e. $2\pi e^{-4}$

$$\iint e^{-x^2-y^2} dx\,dy = \int_0^{2\pi} \int_0^2 e^{-r^2} r\,dr\,d\theta = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^2 = \pi(1 - e^{-4})$$

9. Find the mass of **2 loops** of the helical ramp parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4\theta) \quad \text{for} \quad r \le 3$$

if the density is $\rho = \sqrt{x^2+y^2}$.



a. 40π

b. 120π

c. 200π

d. $\dfrac{500}{3}\pi$

e. $\dfrac{244}{3}\pi$ Correct Choice

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0\ ) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta & 4\ ) \end{vmatrix}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(4\sin\theta) - \hat{\jmath}(4\cos\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$
$$= (4\sin\theta, -4\cos\theta, r)$$
$$\left|\vec{N}\right| = \sqrt{16\sin^2\theta + 16\cos^2\theta + 1} = \sqrt{16+r^2} \qquad \rho = r$$

$$A = \iint \rho\, dS = \iint \rho \left|\vec{N}\right| dr\, d\theta = \int_0^{4\pi}\int_0^3 r\sqrt{16+r^2}\, dr\, d\theta = \frac{4\pi}{3}(16+r^2)^{3/2}\Big|_0^3 = \frac{4\pi}{3}(125-64) = \frac{244}{3}\pi$$

10. Find the flux of $\vec{F} = (y, -x, 2)$ through the helical ramp of problem 9 oriented up.

a. 4π

b. $\dfrac{8}{3}\pi$

c. 216π

d. 108π Correct Choice

e. $\dfrac{1024}{3}\pi$

$$\vec{F} = (y, -x, 1) = (r\sin\theta, -r\cos\theta, 2) \qquad \vec{F}\cdot\vec{N} = 4r\sin^2\theta + 4r\cos^2\theta + 2r = 6r$$

$$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{N}\, dr\, d\theta = \int_0^{4\pi}\int_0^3 6r\, dr\, d\theta = 4\pi[3r^2]_0^3 = 108\pi$$

11. Compute $\int_{(2,1)}^{(3,2)} \vec{F}\cdot d\vec{s}$ for $\vec{F} = (2xy, x^2)$ along the curve $\vec{r}(t) = ((2+t^2)e^{\sin\pi t}, (1+t^2)e^{\sin 2\pi t})$.

HINT: Find a scalar potential.

a. 12

b. 14 Correct Choice

c. 22

d. $\sqrt{2}$

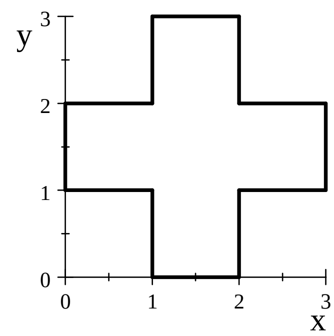
e. $15 - 4\sqrt{2}$

$\vec{F} = (2xy, x^2) = \vec{\nabla} f$ for $f = x^2 y$ $\qquad \int_{(2,1)}^{(3,2)} \vec{F}\cdot d\vec{s} = \int_{(2,1)}^{(3,2)} \vec{\nabla} f\cdot d\vec{s} = f(3,2) - f(2,1) = 3^2 2 - 2^2 1 = 14$

12. Compute $\oint (2x\sin y - 5y)\,dx + (x^2\cos y - 4x)\,dy$

counterclockwise around the cross shown.

HINT: Use Green's Theorem.



a. -45

b. -10

c. 5 Correct Choice

d. 10

e. 45

Green's Theorem says $\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Here $P = 2x\sin y - 5y$ and $Q = x^2\cos y - 4x$.

So $\partial_x Q - \partial_y P = (2x\cos y - 4) - (2x\cos y - 5) = 1$.

So $\iint_R (\partial_x Q - \partial_y P)\,dx\,dy = \iint_R 1\,dx\,dy =$ area $= 5$

13. Compute $\iint_S \vec\nabla \times \vec F \cdot d\vec S$ for $\vec F = (-yz, xz, xyz)$

over the quartic surface $z = (x^2 + y^2)^2$ for $z \le 16$

oriented down and out. The surface may be parametrized by

$$\vec R(r,\theta) = (r\cos\theta, r\sin\theta, r^4)$$

HINT: Use Stokes' Theorem.

a. -128π Correct Choice

b. -64π

c. -32π

d. 32π

e. 64π

Stokes' Theorem says $\iint_S \vec\nabla \times \vec F \cdot d\vec S = \oint_{\partial R} \vec F \cdot d\vec s$.

The boundary is the circle $x^2 + y^2 = 4$ with $z = 16$ which may be parametrized by $\vec r(\theta) = (2\cos\theta, 2\sin\theta, 16)$. The velocity is $\vec v = (-2\sin\theta, 2\cos\theta, 0)$.

Since the surface is oriented down and out, the circle must be traversed clockwise.

So reverse the velocity: $\vec v = (2\sin\theta, -2\cos\theta, 0)$.

$\vec F = (-yz, xz, xyz)$ $\quad \vec F(\vec r(\theta)) = (-32\sin\theta, 32\sin\theta, 64\cos\theta\sin\theta)$

$\vec F \cdot \vec v = -64\sin^2\theta - 64\cos^2\theta = -64$

$\oint_{\partial R} \vec F \cdot d\vec s = \int_0^{2\pi} \vec F \cdot \vec v\,d\theta = -\int_0^{2\pi} 64\,d\theta = -128\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (21 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = (4xz^3, 4yz^3, z^4)$ and the solid V

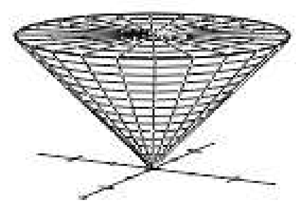
above the cone C given by $z = \sqrt{x^2 + y^2}$

or parametrized by $R(r,\theta) = (r\cos\theta, r\sin\theta, r)$,

below the disk D given by $x^2 + y^2 \le 9$ and $z = 3$.

Be sure to check and explain the orientations.

Use the following steps:



a. (4 pts) Compute the volume integral by successively finding:

$$\vec{\nabla} \cdot \vec{F}, \quad dV, \quad \iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV$$

$$\vec{\nabla} \cdot \vec{F} = 4z^3 + 4z^3 + 4z^3 = 12z^3 \qquad dV = r\,dr\,d\theta\,dz$$

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^3\int_r^3 12z^3\, r\,dz\,dr\,d\theta = 2\pi\int_0^3 [3z^4]_{z=r}^3\, r\,dr = 2\pi\int_0^3 (3^5 - 3r^4)\, r\,dr$$

$$= 2\pi\left[3^5\frac{r^2}{2} - 3\frac{r^6}{6}\right]_{r=0}^3 = 2\pi\left(\frac{3^7}{2} - \frac{3^6}{2}\right) = \pi 3^6(3-1) = 1458\pi$$

b. (8 pts) Compute the surface integral over the disk by parametrizing the disk and successively finding:

$$\vec{R}(r,\theta), \quad \vec{e}_r, \quad \vec{e}_\theta, \quad \vec{N}, \quad \vec{F}\left(\vec{R}(r,\theta)\right), \quad \iint\limits_D \vec{F} \cdot d\vec{S}$$

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3)$$

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{vmatrix}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0,0,r)$$

We need $\vec{N}$ to point up which it does.

$$\vec{F} = (4xz^3, 4yz^3, z^4) \qquad \vec{F}\left(\vec{R}(r,\theta)\right) = (4r\cos\theta\, 27, 4r\sin\theta\, 27, 81)$$

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot \vec{N}\, dr\,d\theta = \int_0^{2\pi}\int_0^3 81r\,dr\,d\theta = 2\pi\left[81\frac{r^2}{2}\right]_0^3 = 729\pi$$

Recall: $\vec{F} = (4xz^3, 4yz^3, z^4)$ and C is the cone parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

c. (7 pts) Compute the surface integral over the cone C by successively finding:

$$\vec{e}_r,\ \ \vec{e}_\theta,\ \ \vec{N},\ \ \vec{F}\big(\vec{R}(r,\theta)\big),\ \ \iint_C \vec{F}\cdot d\vec{S}$$

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

We need $\vec{N}$ to point down (out of the volume). Reverse $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

$$\vec{F} = (4xz^3, 4yz^3, z^4) \qquad \vec{F}\big(\vec{R}(r,\theta)\big) = (4r\cos\theta\, r^3, 4r\sin\theta\, r^3, r^4)$$

$$\vec{F}\cdot\vec{N} = 4r^5\cos^2\theta + 4r^5\sin^2\theta - r^5 = 3r^5$$

$$\iint_P \vec{F}\cdot d\vec{S} = \iint_P \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 3r^5\,dr\,d\theta = 2\pi\left[3\frac{r^6}{6}\right]_0^3 = 2\pi\left(\frac{3^6}{2}\right) = 729\pi$$

d. (2 pts) Combine $\iint_D \vec{F}\cdot d\vec{S}$ and $\iint_C \vec{F}\cdot d\vec{S}$ to get $\iint_{\partial V} \vec{F}\cdot d\vec{S}$

$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_C \vec{F}\cdot d\vec{S} = 729\pi + 729\pi = 1458\pi$$

which agrees with part (a).

15. (10 points) Find the average value of the function $f(x,y,z) = x^2 + y^2 + z^2$ within the solid cylinder $x^2 + y^2 \le 9$ for $0 \le z \le 4$

$$f_{\text{ave}} = \frac{\iiint f\,dV}{\iiint 1\,dV} \qquad \iiint 1\,dV = \int_0^4 \int_0^{2\pi} \int_0^3 r\,dr\,d\theta\,dz = 36\pi = \pi R^2 H$$

$$\iiint f\,dV = \int_0^4 \int_0^{2\pi} \int_0^3 (r^2 + z^2) r\,dr\,d\theta\,dz = 2\pi \int_0^4 \left[\frac{r^4}{4} + z^2 \frac{r^2}{2} \right]_{r=0}^{3} dz = 2\pi \int_0^4 \left(\frac{81}{4} + \frac{9}{2} z^2 \right) dz$$

$$= 2\pi \left[\frac{81}{4} z + \frac{9}{2} \frac{z^3}{3} \right]_{z=0}^{4} = 2\pi(81 + 96) = 354\pi$$

$$f_{\text{ave}} = \frac{354\pi}{36\pi} = \frac{59}{6}$$

16. (10 points) Find the value(s) of R so that the ellipsoid $\frac{x^2}{4^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = R^2$ is tangent to the plane $\frac{1}{2}x + \frac{4}{3}y + 2z = 36$.

HINT: Their normal vectors must be parallel.

Let $f = \frac{x^2}{4^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2}$ and $g = \frac{1}{2}x + \frac{4}{3}y + 2z$.

$$\vec{\nabla} f = \left(\frac{2x}{4^2}, \frac{2y}{3^2}, \frac{2z}{2^2} \right) \qquad \vec{\nabla} g = \left(\frac{1}{2}, \frac{4}{3}, 2 \right) \qquad \vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\frac{2x}{4^2} = \lambda \frac{1}{2} \qquad \frac{2y}{3^2} = \lambda \frac{4}{3} \qquad \frac{2z}{2^2} = \lambda 2$$

$$x = 4\lambda \qquad y = 6\lambda \qquad z = 4\lambda$$

$$36 = \frac{1}{2}x + \frac{4}{3}y + 2z = \frac{1}{2} \cdot 4\lambda + \frac{4}{3} \cdot 6\lambda + 2 \cdot 4\lambda = 18\lambda \quad \Rightarrow \quad \lambda = 2$$

$$x = 8 \qquad y = 12 \qquad z = 8$$

$$R^2 = \frac{x^2}{4^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = \frac{8^2}{4^2} + \frac{12^2}{3^2} + \frac{8^2}{2^2} = 4 + 16 + 16 = 36 \Rightarrow R = 6$$