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MATH 251 Exam 1 Version H Fall 2017

Sections 200 Solutions P. Yasskin

1-9	/54	11	/16
10	/33	Total	/103

Multiple Choice: (6 points each. No part credit.)

1. The points $A = (2,-3.4)$ and $B = (4,1,0)$ are the endpoints of the diameter of a sphere. What is the radius of the sphere?

a. 6

b. 5

c. 4

d. 3 Correct Choice

e. 2

Solution: The diameter is $d = d(A,B) = \sqrt{(4-2)^2 + (1--3)^2 + (0-4)^2} = \sqrt{4+16+16} = 6$. The radius is $r = 3$.

2. Consider the permutation $p = (2,6,4,1,3,5)$. Find its inverse and parity (odd vs. even).

a. $\bar{p} = (4,1,5,3,6,2) \qquad \varepsilon_p = 1$

b. $\bar{p} = (4,1,5,3,6,2) \qquad \varepsilon_p = -1$ Correct Choice

c. $\bar{p} = (2,6,3,5,1,4) \qquad \varepsilon_p = 1$

d. $\bar{p} = (2,6,3,5,1,4) \qquad \varepsilon_p = -1$

e. $\bar{p} = (5,3,1,4,6,2) \qquad \varepsilon_p = 1$

Solution: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 3 & 5 \end{pmatrix} \overset{\text{3 trans}}{\longrightarrow} \begin{pmatrix} 4 & 1 & 2 & 3 & 5 & 6 \\ 1 & 2 & 6 & 4 & 3 & 5 \end{pmatrix} \overset{\text{2 trans}}{\longrightarrow} \begin{pmatrix} 4 & 1 & 5 & 2 & 3 & 6 \\ 1 & 2 & 3 & 6 & 4 & 5 \end{pmatrix}$

$\overset{\text{1 trans}}{\longrightarrow} \begin{pmatrix} 4 & 1 & 5 & 3 & 2 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix} \overset{\text{1 trans}}{\longrightarrow} \begin{pmatrix} 4 & 1 & 5 & 3 & 6 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ 7 transpositions $\Rightarrow$ odd

3. Find the angle between the normals to the planes $3x + 2y - 4z = 3$ and $2x - y + z = 2$.

a. $0°$

b. $30°$

c. $45°$

d. $60°$

e. $90°$ Correct Choice

Solution: The normals are $\vec{N}_1 = (3,2,-4)$ and $\vec{N}_2 = (2,-1,1)$.

Since $\vec{N}_1 \cdot \vec{N}_2 = 6 - 2 - 4 = 0$, the vectors are perpendicular.

4. Duke Skywater pushes an asteroid from the point $P = (2,-3,5)$ to the point $Q = (5,-1,4)$ by the force $\vec{F} = (4,1,2)$. Find the work done to move the asteroid.

a. 16

b. 12 Correct Choice

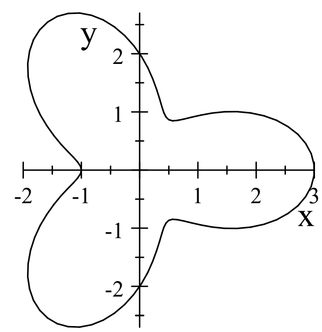
c. 6

d. 4

e. 2

Solution: The displacement is $\overrightarrow{PQ} = Q - P = (3,2,-1)$. So the work done is $W = \vec{F} \cdot \overrightarrow{PQ} = 12 + 2 - 2 = 12$

5. The plot at the right is which polar equation?



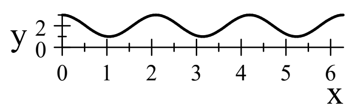
a. $r = 2 + \cos 3\theta$ Correct Choice

b. $r = 2 - \cos 3\theta$

c. $r = 1 + 2\cos 3\theta$

d. $r = 1 - 2\cos 3\theta$

e. $r = 1 + \cos 3\theta$

Solution: The rectangular plot is 

Further, $r = 3$ when $\theta = 0$ and r never goes negative.

6. In $\mathbb{R}^4$, find a vector perpendicular to the hyperplane containing the 4 points
$P = (2,1,4,1)$, $Q = (-1,3,2,1)$, $R = (3,1,2,2)$ and $S = (3,1,4,1)$

a. $(-1,2,2,-4)$

b. $(1,2,2,4)$

c. $(0,2,2,4)$ Correct Choice

d. $(1,-2,2,-4)$

e. $(0,-2,2,-4)$

Solution: $\overrightarrow{PQ} = Q - P = (-3,2,-2,0)$ $\quad \overrightarrow{PR} = R - P = (1,0,-2,1)$ $\quad \overrightarrow{PS} = S - P = (1,0,0,0)$

$$\vec{N} = \perp\left(\overrightarrow{PQ},\overrightarrow{PR},\overrightarrow{PS}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{l} \\ -3 & 2 & -2 & 0 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -3 & -2 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} - \hat{l}\begin{vmatrix} -3 & 2 & -2 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$$

$= (0,2,2,4)$ or any multiple.

7. If $|\vec{u}| = 2$ and $|\vec{v}| = 5$ and $\vec{u} \cdot \vec{v} = 6$ find $|\vec{u} \times \vec{v}|$.

a. 8 Correct Choice

b. 6

c. 4

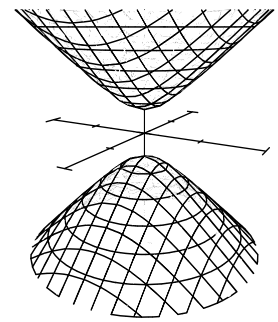
d. 2

e. 0

Solution: By the Pythagorean Identity for Dot and Cross Products, we have

$$|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2} = \sqrt{100 - 36} = 8$$

8. The plot at the right is the graph of which equation?



a. $x^2 + y^2 - z^2 = 1$

b. $x^2 + y^2 - z^2 = 0$

c. $x^2 + y^2 - z^2 = -1$ Correct Choice

d. $x^2 + y^2 - z = 1$

e. $x^2 + y^2 - z = -1$

Solution: (e) is $x^2 + y^2 + 1 = z^2$ So $z \ge 1$ or $z \le -1$.

9. Find the point where the line $(x,y,z) = \vec{r}(t) = (2t+1, t-1, 2t-1)$ intersects the plane $3x + 2y + z = 20$. At this point $x + y + z =$

a. -6

b. -1

c. 4

d. 9 Correct Choice

e. 13

Solution: Plug the line into the plane and solve for t:

$$3x + 2y + z = 3(2t+1) + 2(t-1) + (2t-1) = 10t = 20 \qquad \Rightarrow \qquad t = 2$$

So the point is $(x,y,z) = \vec{r}(2) = (5,1,3)$ and so $x + y + z = 9$.

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (33 points) For the parametric curve $\vec{r}(t) = \left(\frac{2}{t}, 6t, 3t^3\right)$ compute each of the following:

a. (3 pts) velocity $\vec{v}$

Solution: $\vec{v} = \underline{\left(\frac{-2}{t^2}, 6, 9t^2\right)}$

b. (3 pts) acceleration $\vec{a}$

Solution: $\vec{a} = \underline{\left(\frac{4}{t^3}, 0, 18t\right)}$

c. (3 pts) jerk $\vec{j}$

Solution: $\vec{j} = \underline{\left(\frac{-12}{t^4}, 0, 18\right)}$

d. (3 pts) speed $|\vec{v}|$ (Simplify!)
HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{\frac{4}{t^4} + 36 + 81t^4} = \sqrt{\left(\frac{2}{t^2} + 9t^2\right)^2}$ $|\vec{v}| = \underline{\frac{2}{t^2} + 9t^2}$

e. (3 pts) tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\frac{2}{t^2} + 9t^2\right)$ $a_T = \underline{\frac{-4}{t^3} + 18t}$

f. (4 pts) unit binormal $\hat{B}$ (Do this last.)

Solution: $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{-2}{t^2} & 6 & 9t^2 \\ \frac{4}{t^3} & 0 & 18t \end{vmatrix} = \hat{\imath}(108t) - \hat{\jmath}\left(\frac{-36}{t} - \frac{36}{t}\right) + \hat{k}\left(\frac{-24}{t^3}\right)$$
$$= \left(108t, \frac{72}{t}, \frac{-24}{t^3}\right) = 12\left(9t, \frac{6}{t}, \frac{-2}{t^3}\right)$$

$$|\vec{v} \times \vec{a}| = 12\sqrt{81t^2 + \frac{36}{t^2} + \frac{4}{t^6}} = 12\left(9t + \frac{2}{t^3}\right) = \frac{12(9t^4 + 2)}{t^3}$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{t^3}{9t^4 + 2}\left(9t, \frac{6}{t}, \frac{-2}{t^3}\right)$$ $\hat{B} = \underline{\left(\frac{9t^4}{9t^4 + 2}, \frac{6t^2}{9t^4 + 2}, \frac{-2}{9t^4 + 2}\right)}$

Recall: $\vec{r}(t) = \left(\frac{2}{t}, 6t, 3t^3\right)$

g. (2 pts) the values of t where the curve passes thru the points

$A = (2,6,3)$ $\qquad t = $ 1

$B = (1,12,24)$ $\qquad t = $ 2

Solution: Compare each point to the curve $\left(\frac{2}{t}, 6t, 3t^3\right)$. The x component is sufficient, but you should check the other components.

h. (4 pts) arc length between $(2,6,3)$ and $(1,12,24)$, $L = \int_{(2,6,3)}^{(1,12,24)} ds$

Solution: $L = \int_1^2 |\vec{v}|\,dt = \int_1^2 \left(\frac{2}{t^2} + 9t^2\right) dt = \left[\frac{-2}{t} + 3t^3\right]_1^2 = (-1+24) - (-2+3)$

$L = $ 22

i. (4 pts) A wire has the shape of this curve between $(2,6,3)$ and $(1,12,24)$. Find the mass of the wire if the linear mass density is $\rho = \frac{1}{6}xz$.
(Don't simplify the answer.)

Solution: $\vec{v} = \left(\frac{-2}{t^2}, 6, 9t^2\right)$ $\qquad |\vec{v}| = \frac{2}{t^2} + 9t^2$ $\qquad \rho = \frac{1}{6}xz = \frac{1}{6}\left(\frac{2}{t}\right)(3t^3) = t^2$

$M = \int_{(2,6,3)}^{(1,12,12)} \rho\,ds = \int_1^2 \frac{1}{6}xz|\vec{v}|\,dt = \int_1^2 t^2\left(\frac{2}{t^2} + 9t^2\right) dt = \int_1^2 (2 + 9t^4)\,dt = \left[2t + \frac{9t^5}{5}\right]_1^2$

$M = \left(4 + \frac{9 \cdot 2^5}{5}\right) - \left(2 + \frac{9}{5}\right) = \frac{289}{5}$

j. (4 pts) A wire has the shape of this curve. Find the work done by the force $\vec{F} = (z,y,x)$ which pushes a bead along the wire from $(2,6,3)$ to $(1,12,24)$.

Solution: $\vec{F} = (z,y,x) = \left(3t^3, 6t, \frac{2}{t}\right)$ $\quad \vec{v} = \left(\frac{-2}{t^2}, 6, 9t^2\right)$

$\vec{F} \cdot \vec{v} = 3t^3 \frac{-2}{t^2} + 6t6 + \frac{2}{t}9t^2 = -6t + 36t + 18t = 48t$

$W = \int_{(2,6,3)}^{(1,12,12)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v}\,dt = \int_1^2 48t\,dt = \left[24t^2\right]_1^2 = 24(4-1)$

$W = $ 72

11. (16 points) Are the following lines parallel, intersecting or skew? If they intersect, find the point of intersection.

a. Line 1: $\vec{r}_1(t) = (t+2, t-2, 2t-1)$

Line 2: $\vec{r}_2(t) = (t+1, 2t-6, 2t-1)$

Solution: The direction vectors, $\vec{v}_1 = (1,1,2)$ and $\vec{v}_2 = (1,2,2)$, are not multiples of each other. So the lines are not parallel. Since the parameter values may be different at the intersection point, we rewrite the second line as $\vec{r}_2(s) = (s+1, 2s-6, 2s-1)$. We set the x and y components equal to find where the projections intersect in the xy-plane:

$$t+2 = s+1 \qquad t-2 = 2s-6$$

The first equation says $s = t+1$. So the second equation says $t-2 = 2(t+1)-6 = 2t-4$. Or $t = 2$ and so $s = 3$. So the points are

$$\vec{r}_1(2) = (4,0,3) \qquad \vec{r}_2(3) = (4,0,5)$$

They do not intersect. They are skew.

b. Line 1: $\vec{r}_1(t) = (t+2, t-2, 2t+1)$

Line 2: $\vec{r}_2(t) = (t+1, 2t-6, 2t-1)$

Solution: The direction vectors, $\vec{v}_1 = (1,1,2)$ and $\vec{v}_2 = (1,2,2)$, are not multiples of each other. So the lines are not parallel. Since the parameter values may be different at the intersection point, we rewrite the second line as $\vec{r}_2(s) = (s+1, 2s-6, 2s-1)$. We set the x and y components equal to find where the projections intersect in the xy-plane::

$$t+2 = s+1 \qquad t-2 = 2s-6$$

The first equation says $s = t+1$. So the second equation says $t-2 = 2(t+1)-6 = 2t-4$. Or $t = 2$ and so $s = 3$. So the points are

$$\vec{r}_1(2) = (4,0,5) \qquad \vec{r}_2(3) = (4,0,5)$$

They intersect at $(4,0,5)$.