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MATH 251 Exam 2 Version B Fall 2017

Sections 515 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1-13	/65	15	/15
14	/15	16	/15
		Total	/110

1. Find the equation of the plane tangent to $z = x^2y + xy^2$ at $(x,y) = (1,2)$.
The z -intercept is:

- a. $c = -24$
- b. $c = -12$ Correct Choice
- c. $c = 12$
- d. $c = -6$
- e. $c = 6$

Solution: $a = 1$ and $b = 2$.

$$\begin{aligned} f &= x^2y + xy^2 & f(1,2) &= 6 \\ f_x &= 2xy + y^2 & f_x(1,2) &= 8 \\ f_y &= x^2 + 2xy & f_y(1,2) &= 5 \end{aligned}$$

So the tangent plane is

$$\begin{aligned} z &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 6 + 8(x-1) + 5(y-2) \\ &= 8x + 5y - 12 \end{aligned}$$

So the z -intercept is $c = -12$.

2. Find the plane tangent to the ellipsoid $36x^2 + 9y^2 + 4z^2 = 108$ at the point $(x,y,z) = (1,2,3)$.

- a. $6x + 12y + 18z = 84$
- b. $\frac{x}{6} + \frac{y}{12} + \frac{z}{18} = \frac{1}{2}$
- c. $6x + 3y + 2z = 18$ Correct Choice
- d. $\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = \frac{7}{3}$
- e. $36x + 9y + 4z = 18$

Solution: $F = 36x^2 + 9y^2 + 4z^2$ $\vec{\nabla}F = \langle 72x, 18y, 8z \rangle$ $\vec{N} = \vec{\nabla}F|_{(1,2,3)} = \langle 72, 36, 24 \rangle$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ which is

$$\begin{aligned} \langle 72, 36, 24 \rangle \cdot (x,y,z) &= \langle 72, 36, 24 \rangle \cdot (1,2,3) \\ 72x + 36y + 24z &= 216 \\ 6x + 3y + 2z &= 18 \end{aligned}$$

3. If $f(x,y) = x \cos(y) + y \sin(x)$, which of the following is INCORRECT?

- a. $f_x = \cos(y) + y \cos(x)$
- b. $f_y = -x \sin(y) + \sin(x)$
- c. $f_{xx} = -y \sin(x)$
- d. $f_{xy} = -\sin(y) + \cos(x)$
- e. $f_{yx} = \sin(y) + \cos(x)$ Correct Choice

Solution: $f_{xy} = f_{yx}$ So one of those must be wrong. It's f_{yx} because $[\cos(y)]' = -\sin(y)$.

4. A support beam is constructed using four struts whose lengths are w , x , y and z . The strength of the beam is $S = w^2x + y^2z$. If the current lengths are $w = 1$, $x = 3$, $y = 2$ and $z = 1$, then the current strength is $S = 1^2 \cdot 3 + 2^2 \cdot 1 = 7$. Use differentials (i.e. the linear approximation) to estimate how much the strength increases, ΔS , if the lengths increase by $\Delta w = 0.1$, $\Delta x = 0.2$, $\Delta y = 0.2$ and $\Delta z = 0.3$.

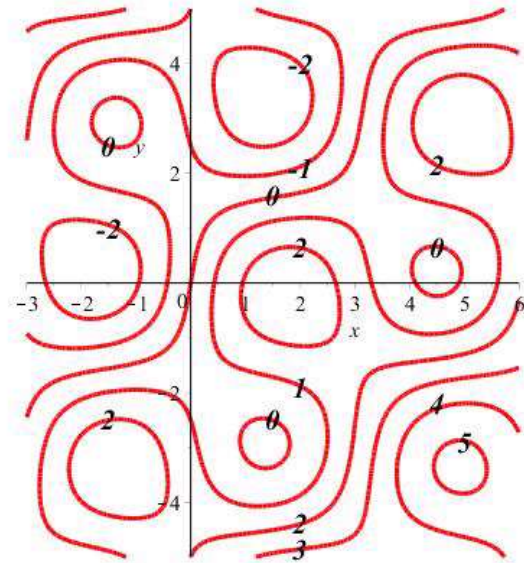
- a. 0.8
- b. 1.4
- c. 2.1
- d. 2.8 Correct Choice
- e. 3.5

Solution:

$$\begin{aligned} \Delta S \approx dS &= \frac{\partial S}{\partial w} dw + \frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial z} dz \\ &= 2wx dw + w^2 dx + 2yz dy + y^2 dz \\ &= 6(0.1) + 1(0.2) + 4(0.2) + 4(0.3) = .6 + .2 + .8 + 1.2 = 2.8 \end{aligned}$$

5. In the contour plot at the right, which point is the saddle point?

- a. (3.5, 1.5) Correct Choice
- b. (5, -1)
- c. (5, -3.5)
- d. (1.5, 3.5)
- e. (-1.5, -3.5)



Solution: (a), (d) and (e) are in the middle of "circles", so they are maxima or minima. (b) is in "parallel lines", so it's a sloped area. (c) is in the middle of "hyperbolas", so it is the saddle point.

6. Use the linear approximation to the function $f(x,y) = \sqrt{x^2 + y^2}$ to estimate $\sqrt{3.9^2 + 3.2^2}$.

- a. 5.02
- b. 5.04 Correct Choice
- c. 5.10
- d. 5.40
- e. 5.73

Solution: $f(x,y) = \sqrt{x^2 + y^2}$ $f(4,3) = \sqrt{4^2 + 3^2} = 5$

$$f_x(x,y) = \frac{x}{\sqrt{x^2 + y^2}} \qquad f_x(4,3) = \frac{4}{\sqrt{4^2 + 3^2}} = \frac{4}{5}$$
$$f_y(x,y) = \frac{y}{\sqrt{x^2 + y^2}} \qquad f_y(4,3) = \frac{3}{\sqrt{4^2 + 3^2}} = \frac{3}{5}$$

$$f_{\tan}(x,y) = f(4,3) + f_x(4,3)(x - 4) + f_y(4,3)(y - 3) = 5 + \frac{4}{5}(x - 4) + \frac{3}{5}(y - 3)$$

$$\sqrt{3.9^2 + 3.2^2} = f(3.9,3.2) \approx f_{\tan}(3.9,3.2) = 5 + .8(-.1) + .6(.2) = 5.04$$

7. A weather balloon is currently located at $(x,y,z) = (20,30,10)$ and has velocity $\vec{v} = (3,1,2)$. At the current time, it measures that the pressure is $P = .96$ atm and has gradient

$$\vec{\nabla}P = \left\langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right\rangle = \langle .01, .02, .03 \rangle$$

Find the rate of change of the pressure as seen aboard the balloon.

- a. 0.08
- b. 0.09
- c. 0.10
- d. 0.11 Correct Choice
- e. 0.12

Solution: $\frac{dP}{dt} = \vec{v} \cdot \vec{\nabla}P = 3(.01) + 1(.02) + 2(.03) = 0.11$

8. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = xyz$. If Ham's current position is $P = (1, 1, 2)$, find the rate of change of the density in the direction toward the point $Q = (-1, 3, 3)$.
- $\frac{5}{3}$
 - $\frac{4}{3}$
 - 1
 - $\frac{2}{3}$
 - $\frac{1}{3}$ Correct Choice

Solution: Since we want the direction toward Q , we need the directional derivative of δ using a unit vector. The vector from P to Q is $\overrightarrow{PQ} = Q - P = (-2, 2, 1)$. Its magnitude and direction are

$$|\overrightarrow{PQ}| = \sqrt{4 + 4 + 1} = 3 \quad \widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

The gradient of the density is $\vec{\nabla}\delta = \langle yz, xz, xy \rangle$. At P this is $\vec{\nabla}\delta|_P = \langle 2, 2, 1 \rangle$. So the directional derivative is

$$\begin{aligned} \nabla_{\widehat{PQ}}\delta &= \widehat{PQ} \cdot \vec{\nabla}\delta = \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle \cdot \langle 2, 2, 1 \rangle \\ &= \frac{1}{3}(-4 + 4 + 1) = \frac{1}{3} \end{aligned}$$

9. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = xyz$. If Ham's current position is $P = (1, 1, 2)$, in what unit vector direction should he travel to increase the cloaking sparkles as fast as possible?
- $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ Correct Choice
 - $\left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$
 - $\left\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$
 - $\left\langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$
 - $\left\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

Solution: The direction of maximum increase is the direction of the gradient.

$$\vec{\nabla}\delta = \langle yz, xz, xy \rangle \quad \vec{\nabla}\delta|_P = \langle 2, 2, 1 \rangle \quad |\vec{\nabla}\delta| = \sqrt{4 + 4 + 1} = 3$$

$$\hat{u} = \frac{\vec{\nabla}\delta}{|\vec{\nabla}\delta|} = \frac{1}{3}\langle 2, 2, 1 \rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

10. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \cdot \vec{F} =$

- a. $-y^2 - z^2 - x^2$
- b. $2xy - 2yz + 2zx$
- c. $2xy + 2yz + 2zx$ Correct Choice
- d. $\langle 2xy, -2yz, 2zx \rangle$
- e. $\langle 2xy, 2yz, 2zx \rangle$

Solution: $\vec{\nabla} \cdot \vec{F} = \partial_x(x^2y) + \partial_y(y^2z) + \partial_z(z^2x) = 2xy + 2yz + 2zx$

11. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \times \vec{F} =$

- a. $-y^2 + z^2 - x^2$
- b. $\langle 2xy, -2yz, 2zx \rangle$
- c. $\langle 2xy, 2yz, 2zx \rangle$
- d. $\langle -y^2, -z^2, -x^2 \rangle$ Correct Choice
- e. $\langle -y^2, z^2, -x^2 \rangle$

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & y^2z & z^2x \end{vmatrix} = \hat{i}(0 - y^2) - \hat{j}(z^2 - 0) + \hat{k}(0 - x^2) = \langle -y^2, -z^2, -x^2 \rangle$

12. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} =$

- a. $-y^2 + z^2 - x^2$
- b. $-y^2 - z^2 - x^2$
- c. $2y + 2z + 2x$
- d. $2y - 2z + 2x$
- e. 0 Correct Choice

Solution: $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any vector field with continuous second derivatives.

13. Find a scalar potential, f , for the vector field $\vec{F} = \langle yz + 6x, xz - 4y, xy \rangle$.

Then $f(2,2,2) - f(1,1,1) =$

- a. 15
- b. 10 Correct Choice
- c. 5
- d. 2
- e. 1

Solution: To solve $\vec{\nabla}f = F$, we must solve $\partial_x f = yz + 6x$, $\partial_y f = xz - 4y$, $\partial_z f = xy$.

The first says $f = xyz + 3x^2 + g(y,z)$. Then the second says $\partial_y f = xz + \partial_y g = xz - 4y$.

So $g = -2y^2 + h(z)$, or $f = xyz + 3x^2 - 2y^2 + h(z)$. Then the third says $\partial_z f = xy + h'(z) = xy$.

So $h = C$. Consequently, $f = xyz + 3x^2 - 2y^2 + C$.

$$f(2,2,2) - f(1,1,1) = (8 + 12 - 8) - (1 + 3 - 2) = 10$$

Work Out: (15 points each. Part credit possible. Show all work.)

14. (15 points) The Ideal Gas Law says the Pressure, P , Volume, V , and Temperature, T , are related by $PV = kT$. Currently, a particular sample of ideal gas has the parameters:

$$P = 0.9 \text{ atm} \qquad V = 600 \text{ cm}^3 \qquad \text{and} \qquad T = 270^\circ\text{K}$$

a. First find the constant k .

Solution: Using current values, we find $k = \frac{PV}{T} = \frac{(.9)(600)}{270} = 2$.

b. If the volume is increasing at $\frac{dV}{dt} = \frac{8 \text{ cm}^3}{\text{hr}}$ while the temperature is increasing

at $\frac{dT}{dt} = \frac{3^\circ\text{K}}{\text{hr}}$, at what rate, $\frac{dP}{dt}$, is the pressure changing?

Is the pressure increasing or decreasing?

Solution: Using $k = 2$, we have $P = \frac{2T}{V}$. We apply the chain rule:

$$\frac{dP}{dt} = \frac{\partial P}{\partial V} \frac{dV}{dt} + \frac{\partial P}{\partial T} \frac{dT}{dt} = -\frac{2T}{V^2} \frac{dV}{dt} + \frac{2}{V} \frac{dT}{dt} = -\frac{2 \cdot 270}{600^2} 8 + \frac{2}{600} 3 = -.002$$

The pressure is decreasing.

15. (15 points) Find all critical points of the function $f(x,y) = x^3 - 12x + 3xy^2$.
 Then use the second derivative test to classify each as a
 local minimum, local maximum or saddle or say the test fails.

Solution: $f_x = 3x^2 - 12 + 3y^2 = 0$ $f_y = 6xy = 0$

$f_y = 6xy = 0$ leads to 2 cases:

Case 1: $x = 0$: $f_x = -12 + 3y^2 = 0 \Rightarrow y = \pm 2 \Rightarrow$ critical points: $(0, 2), (0, -2)$

Case 2: $y = 0$: $f_x = 3x^2 - 12 = 0 \Rightarrow x = \pm 2 \Rightarrow$ critical points: $(2, 0), (-2, 0)$

$$f_{xx} = 6x \qquad f_{yy} = 6x \qquad f_{xy} = 6y$$

x	y	f_{xx}	f_{yy}	f_{xy}	D	Classification
0	2	0	0	12	-144	saddle
0	-2	0	0	-12	-144	saddle
2	0	12	12	0	144	local minimum
-2	0	-12	-12	0	144	local maximum

16. (15 points) Find the point on the plane $2x - 2y - z = 18$ that is closest to the origin. You may use either the Eliminate a Variable method or the Lagrange Multiplier method.

Solution 1: Eliminate a Variable Method:

We minimize the square of the distance:

$$f = D^2 = x^2 + y^2 + z^2$$

subject to the constraint $z = 2x - 2y - 18$. We substitute the constraint into f :

$$f = x^2 + y^2 + (2x - 2y - 18)^2$$

We set the x and y derivatives equal to 0 and solve for x and y .

$$f_x = 2x + 4(2x - 2y - 18) = 10x - 8y - 72 = 0$$

$$f_y = 2y - 4(2x - 2y - 18) = -8x + 10y + 72 = 0$$

Equivalently:

$$5x - 4y = 36$$

$$-4x + 5y = -36$$

Multiply the first equation by 4 and the second equation by 5 and add:

$$20x - 16y = 144$$

$$-20x + 25y = -180$$

$$9y = -36 \quad \Rightarrow \quad y = -4$$

We substitute back:

$$5x - 4(-4) = 36 \quad \Rightarrow \quad x = 4$$

$$z = 2(4) - 2(-4) - 18 = -2$$

So the closest point is $(x, y, z) = (4, -4, -2)$.

Solution 2: Lagrange Multiplier Method:

We minimize the square of the distance:

$$f = D^2 = x^2 + y^2 + z^2$$

subject to the constraint $g = 2x - 2y - z = 18$. The gradients are:

$$\vec{\nabla}f = \langle 2x, 2y, 2z \rangle \quad \vec{\nabla}g = \langle 2, -2, -1 \rangle$$

The Lagrange equations $\vec{\nabla}f = \lambda \vec{\nabla}g$ are

$$2x = 2\lambda \quad 2y = -2\lambda \quad 2z = -\lambda$$

Consequently, $\lambda = x = -y = -2z$. We substitute these into the constraint:

$$18 = 2x - 2y - z = -4z - 4z - z = -9z \quad \Rightarrow \quad z = -2$$

So $x = 4$ and $y = -4$. So the closest point is $(x, y, z) = (4, -4, -2)$.