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1-13	/65	15	/15
14	/15	16	/15
		Total	/110

MATH 251 Exam 2 Version H Fall 2017

Sections 200 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the plane tangent to $z = x^2y + xy^2$ at $(x,y) = (1,2)$.
The z-intercept is:

a. $c = -6$

b. $c = 6$

c. $c = -12$ Correct Choice

d. $c = 12$

e. $c = -24$

Solution: $a = 1$ and $b = 2$.

$$f = x^2y + xy^2 \qquad f(1,2) = 6$$
$$f_x = 2xy + y^2 \qquad f_x(1,2) = 8$$
$$f_y = x^2 + 2xy \qquad f_y(1,2) = 5$$

So the tangent plane is

$$\begin{aligned} z &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 6 + 8(x-1) + 5(y-2) \\ &= 8x + 5y - 12 \end{aligned}$$

So the z-intercept is $c = -12$.

2. Find the plane tangent to the ellipsoid $36x^2 + 9y^2 + 4z^2 = 108$ at the point $(x,y,z) = (1,2,3)$.

a. $6x + 3y + 2z = 18$ Correct Choice

b. $\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = \frac{7}{3}$

c. $6x + 12y + 18z = 84$

d. $\frac{x}{6} + \frac{y}{12} + \frac{z}{18} = \frac{1}{2}$

e. $36x + 9y + 4z = 18$

Solution: $F = 36x^2 + 9y^2 + 4z^2 \qquad \vec{\nabla}F = \langle 72x, 18y, 8z \rangle \qquad \vec{N} = \vec{\nabla}F\Big|_{(1,2,3)} = \langle 72, 36, 24 \rangle$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ which is

$$\langle 72,36,24 \rangle \cdot (x,y,z) = \langle 72,36,24 \rangle \cdot (1,2,3)$$
$$72x + 36y + 24z = 216$$
$$6x + 3y + 2z = 18$$

3. If $f(x,y) = x\cos(y) + y\sin(x)$, which of the following is INCORRECT?

a. $f_x = \cos(y) + y\cos(x)$

b. $f_y = -x\sin(y) + \sin(x)$

c. $f_{xx} = -y\sin(x)$

d. $f_{xy} = \sin(y) + \cos(x)$ Correct Choice

e. $f_{yx} = -\sin(y) + \cos(x)$

Solution: $f_{xy} = f_{yx}$ So one of those must be wrong. It's f_{xy} because $[\cos(y)]' = -\sin(y)$.

4. A support beam is constructed using four struts whose lengths are w, x, y and z. The strength of the beam is $S = w^2x + y^2z$. If the current lengths are $w = 1$, $x = 3$, $y = 2$ and $z = 1$, then the current strength is $S = 1^2 3 + 2^2 1 = 7$. Use differentials (i.e. the linear approximation) to estimate how much the strength increases, ΔS, if the lengths increase by $\Delta w = 0.1$, $\Delta x = 0.2$, $\Delta y = 0.2$ and $\Delta z = 0.3$.

a. 3.5

b. 2.8 Correct Choice

c. 2.1

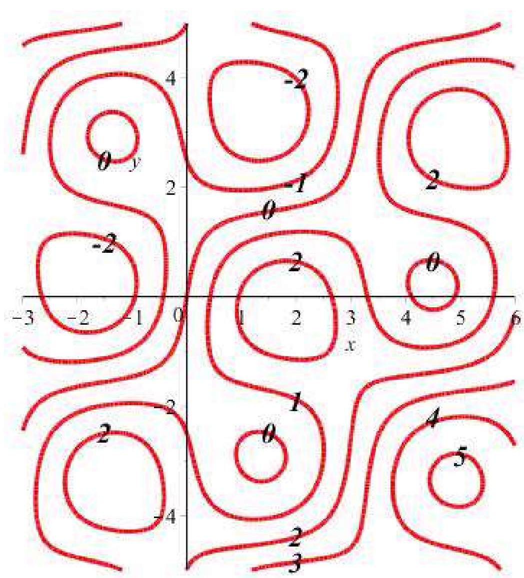
d. 1.4

e. 0.8

Solution:

$$\begin{aligned}\Delta S \approx dS &= \frac{\partial S}{\partial w}dw + \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz \\ &= 2wx\,dw + w^2\,dx + 2yz\,dy + y^2\,dz \\ &= 6(0.1) + 1(0.2) + 4(0.2) + 4(0.3) = .6 + .2 + .8 + 1.2 = 2.8\end{aligned}$$

5. In the coutour plot at the right, which point is the saddle point?



a. $(1.5, 3.5)$

b. $(5, -1)$

c. $(3.5, 1.5)$ Correct Choice

d. $(5, -3.5)$

e. $(-1.5, -3.5)$

Solution: (a), (d) and (e) are in the middle of "circles", so they are maxima or minima.
(b) is in "parallel lines", so it's a sloped area. (c) is in the middle of "hyperbolas", so it is the saddle point.

6. Use the linear approximation to the function $f(x,y) = \sqrt{x^2+y^2}$ to estimate $\sqrt{3.9^2+3.2^2}$.

a. 5.73

b. 5.40

c. 5.10

d. 5.04 Correct Choice

e. 5.02

Solution: $f(x,y) = \sqrt{x^2+y^2}$ $f(4,3) = \sqrt{4^2+3^2} = 5$

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_x(4,3) = \frac{4}{\sqrt{4^2+3^2}} = \frac{4}{5}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} \qquad f_y(4,3) = \frac{3}{\sqrt{4^2+3^2}} = \frac{3}{5}$$

$$f_{\tan}(x,y) = f(4,3) + f_x(4,3)(x-4) + f_y(4,3)(y-3) = 5 + \frac{4}{5}(x-4) + \frac{3}{5}(y-3)$$

$$\sqrt{3.9^2+3.2^2} = f(3.9,3.2) \approx f_{\tan}(3.9,3.2) = 5 + .8(-.1) + .6(.2) = 5.04$$

7. A weather balloon is currently located at $(x,y,z) = (20,30,10)$ and has velocity $\vec{v} = (3,1,2)$. At the current time, it measures that the pressure is $P = .96$ atm and has gradient

$$\vec{\nabla}P = \left\langle \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right\rangle = \langle .01, .02, .03 \rangle$$

Find the rate of change of the pressure as seen aboard the balloon.

a. 0.12

b. 0.11 Correct Choice

c. 0.10

d. 0.09

e. 0.08

Solution: $\dfrac{dP}{dt} = \vec{v} \cdot \vec{\nabla}P = 3(.01) + 1(.02) + 2(.03) = 0.11$

8. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = xyz$. If Ham's current position is $P = (1,1,2)$, find the rate of change of the density in the direction toward the point $Q = (-1,3,3)$.

a. $\frac{1}{3}$ Correct Choice

b. $\frac{2}{3}$

c. 1

d. $\frac{4}{3}$

e. $\frac{5}{3}$

Solution: Since we want the direction toward Q, we need the directional derivative of δ using a unit vector. The vector from P to Q is $\overrightarrow{PQ} = Q - P = (-2,2,1)$. Its magnitude and direction are

$$\left|\overrightarrow{PQ}\right| = \sqrt{4+4+1} = 3 \qquad \widehat{PQ} = \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

The gradient of the density is $\vec{\nabla}\delta = \langle yz, xz, xy\rangle$. At P this is $\vec{\nabla}\delta\big|_P = \langle 2,2,1\rangle$. So the directional derivative is

$$\nabla_{\widehat{PQ}}\delta = \widehat{PQ} \cdot \vec{\nabla}\delta = \left\langle \frac{-2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle \cdot \langle 2,2,1\rangle$$
$$= \frac{1}{3}(-4+4+1) = \frac{1}{3}$$

9. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = xyz$. If Ham's current position is $P = (1,1,2)$, in what unit vector direction should he travel to increase the cloaking sparkles as fast as possible?

a. $\left\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

b. $\left\langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$

c. $\left\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$

d. $\left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$

e. $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$ Correct Choice

Solution: The direction of maximum increase is the direction of the gradient.

$$\vec{\nabla}\delta = \langle yz, xz, xy\rangle \qquad \vec{\nabla}\delta\big|_P = \langle 2,2,1\rangle \qquad \left|\vec{\nabla}\delta\right| = \sqrt{4+4+1} = 3$$

$$\hat{u} = \frac{\vec{\nabla}\delta}{\left|\vec{\nabla}\delta\right|} = \frac{1}{3}\langle 2,2,1\rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

10. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \cdot \vec{F} =$

a. $-y^2 - z^2 - x^2$

b. $2xy + 2yz + 2zx$ Correct Choice

c. $2xy - 2yz + 2zx$

d. $\langle 2xy, 2yz, 2zx \rangle$

e. $\langle 2xy, -2yz, 2zx \rangle$

Solution: $\vec{\nabla} \cdot \vec{F} = \partial_x(x^2y) + \partial_y(y^2z) + \partial_z(z^2x) = 2xy + 2yz + 2zx$

11. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \times \vec{F} =$

a. $-y^2 + z^2 - x^2$

b. $\langle -y^2, z^2, -x^2 \rangle$

c. $\langle -y^2, -z^2, -x^2 \rangle$ Correct Choice

d. $\langle 2xy, 2yz, 2zx \rangle$

e. $\langle 2xy, -2yz, 2zx \rangle$

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & y^2z & z^2x \end{vmatrix} = \hat{\imath}(0 - y^2) - \hat{\jmath}(z^2 - 0) + \hat{k}(0 - x^2) = \langle -y^2, -z^2, -x^2 \rangle$

12. If $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$, then $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} =$

a. $-y^2 - z^2 - x^2$

b. $-y^2 + z^2 - x^2$

c. $2y - 2z + 2x$

d. $2y + 2z + 2x$

e. 0 Correct Choice

Solution: $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any vector field with continuous second derivatives.

13. Find a scalar potential, f, for the vector field $\vec{F} = \langle yz + 6x, xz - 4y, xy \rangle$.
Then $f(2,2,2) - f(1,1,1) =$

a. 1

b. 2

c. 5

d. 10 Correct Choice

e. 15

Solution: To solve $\vec{\nabla} f = F$, we must solve $\partial_x f = yz + 6x$, $\partial_y f = xz - 4y$, $\partial_z f = xy$.
The first says $f = xyz + 3x^2 + g(y,z)$. Then the second says $\partial_y f = xz + \partial_y g = xz - 4y$.
So $g = -2y^2 + h(z)$, or $f = xyz + 3x^2 - 2y^2 + h(z)$. Then the third says $\partial_z f = xy + h'(z) = xy$.
So $h = C$. Consequently, $f = xyz + 3x^2 - 2y^2 + C$.

$$f(2,2,2) - f(1,1,1) = (8 + 12 - 8) - (1 + 3 - 2) = 10$$

Work Out: (15 points each. Part credit possible. Show all work.)

14. (15 points) Find all critical points of the function $f(x,y) = x^3 - 12x + 3xy^2$.
Then use the second derivative test to classify each as a
local minimum, local maximum or saddle or say the test fails.

Solution: $f_x = 3x^2 - 12 + 3y^2 = 0$ $\quad f_y = 6xy = 0$

$f_y = 6xy = 0$ leads to 2 cases:

Case1: $x = 0$: $f_x = -12 + 3y^2 = 0 \Rightarrow y = \pm 2 \Rightarrow$ critical points: $(0,2)$, $(0,-2)$

Case 2: $y = 0$: $f_x = 3x^2 - 12 = 0 \Rightarrow x = \pm 2 \Rightarrow$ critical points: $(2,0)$, $(-2,0)$

$$f_{xx} = 6x \qquad f_{yy} = 6x \qquad f_{xy} = 6y$$

x	y	f_{xx}	f_{yy}	f_{xy}	D	Classification
0	2	0	0	12	-144	saddle
0	-2	0	0	-12	-144	saddle
2	0	12	12	0	144	local minimum
-2	0	-12	-12	0	144	local maximum

15. (15 points) For each limit, either prove the limit does not exist or prove it does exist and give its limit.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

Solution: Switch to polar coordinates: $x = r\cos\theta \qquad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r\cos\theta\sin^2\theta$$

Since $r \to 0$ and $\cos\theta\sin^2\theta$ is bounded, the limit is 0.
More precisely, since $-1 \le \cos\theta\sin^2\theta \le 1$, we have $-r \le r\cos\theta\sin^2\theta \le r$.
Since $\displaystyle\lim_{r\to 0}(-r) = \lim_{r\to 0}(r) = 0$, by the pinching theorem, $\displaystyle\lim_{r\to 0} r\cos\theta\sin^2\theta = 0$ also.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Solution: $$\lim_{\substack{y=mx\\x\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{xm^2x^2}{x^2+m^4x^4} = \lim_{x\to 0} \frac{xm^2}{1+m^4x^2} = \frac{0}{1} = 0$$

$$\lim_{\substack{x=y^2\\x\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2y^2}{y^4+y^4} = \frac{1}{2}$$

Since the limit is different from 2 directions, the limit does not exist.

16. (15 points) Find the point on the plane $2x-2y-z=18$ that is closest to the origin.
You may use either the Eliminate a Variable method or the Lagrange Multiplier method.

Solution 1: Eliminate a Variable Method:

We minimize the square of the distance:

$$f = D^2 = x^2 + y^2 + z^2$$

subject to the constraint $z = 2x - 2y - 18$. We substitute the constraint into f:

$$f = x^2 + y^2 + (2x - 2y - 18)^2$$

We set the x and y derivatives equal to 0 and solve for x and y.

$$f_x = 2x + 4(2x - 2y - 18) = 10x - 8y - 72 = 0$$

$$f_y = 2y - 4(2x - 2y - 18) = -8x + 10y + 72 = 0$$

Equivalently:

$$5x - 4y = 36$$

$$-4x + 5y = -36$$

Multiply the first equation by 4 and the second equation by 5 and add:

$$20x - 16y = 144$$

$$-20x + 25y = -180$$

$$9y = -36 \qquad \Rightarrow \qquad y = -4$$

We substitute back:

$$5x - 4(-4) = 36 \qquad \Rightarrow \qquad x = 4$$

$$z = 2(4) - 2(-4) - 18 = -2$$

So the closest point is $(x,y,z) = (4,-4,-2)$.

Solution 2: Lagrange Multiplier Method:

We minimize the square of the distance:

$$f = D^2 = x^2 + y^2 + z^2$$

subject to the constraint $g = 2x - 2y - z = 18$. The gradients are:

$$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle \qquad \vec{\nabla} g = \langle 2, -2, -1 \rangle$$

The Lagrange equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ are

$$2x = 2\lambda \qquad 2y = -2\lambda \qquad 2z = -\lambda$$

Consequently, $\lambda = x = -y = -2z$. We substitute these into the constraint:

$$18 = 2x - 2y - z = -4z - 4z - z = -9z \qquad \Rightarrow \qquad z = -2$$

So $x = 4$ and $y = -4$. So the closest point is $(x,y,z) = (4,-4,-2)$.