

Name \_\_\_\_\_

MATH 251 Exam 3 Version A Fall 2017

Sections 515 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-10	/60	15	/15
14	/15	16	/15
		Total	/105

1. Compute  $\int_0^2 \int_{x^2}^{2x} 2y \, dy \, dx$ .

- a.  $-\frac{56}{15}$
- b.  $-\frac{32}{15}$
- c.  $\frac{32}{15}$
- d.  $\frac{64}{15}$  Correct Choice
- e.  $\frac{128}{15}$

**Solution:**  $\int_0^2 \int_{x^2}^{2x} 2y \, dy \, dx = \int_0^2 [y^2]_{y=x^2}^{2x} \, dx = \int_0^2 (4x^2 - x^4) \, dx = \left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$

2. Find the average value of the function  $f(x,y) = x^2 \sin y$  on the rectangle  $[0, 3] \times [0, \pi]$ .

- a.  $\frac{18}{\pi}$
- b.  $\frac{6}{\pi}$  Correct Choice
- c.  $3\pi$
- d. 18
- e.  $54\pi$

**Solution:** The area is  $A = 3\pi$ . The integral of  $f$  is

$$\iint f \, dA = \int_0^\pi \int_0^3 x^2 \sin y \, dx \, dy = \left[ \frac{x^3}{3} \right]_0^3 [-\cos y]_0^\pi = 9(-1 - (-1)) = 18$$

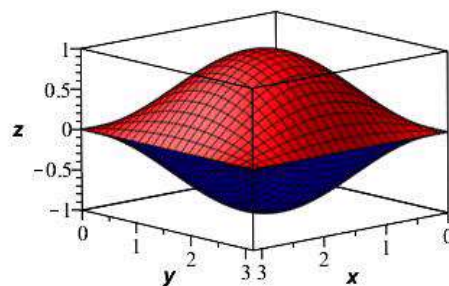
So the average is  $f_{\text{ave}} = \frac{1}{A} \iint f \, dA = \frac{18}{3\pi} = \frac{6}{\pi}$

3. Find the volume of the ravioli bounded by

$$-\sin x \sin y \leq z \leq \sin x \sin y$$

for  $0 \leq x \leq \pi$  and  $0 \leq y \leq \pi$ .

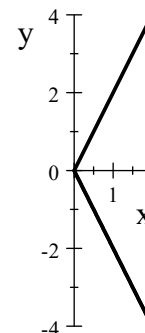
- a. 2
- b. 4
- c. 8 **Correct Choice**
- d.  $\pi^2$
- e.  $4\pi^2$



**Solution:**  $A = \int_0^\pi \int_0^\pi (\sin x \sin y - (-\sin x \sin y)) dx dy = 2 \int_0^\pi \int_0^\pi \sin x \sin y dx dy$   
 $= 2 \int_0^\pi \sin x dx \int_0^\pi \sin y dy = 2[-\cos x]_0^\pi [-\cos y]_0^\pi = 2(2)(2) = 8$

4. Find the mass of a triangular plate with vertices  $(0,0)$ ,  $(2,4)$  and  $(2,-4)$  whose surface mass density is  $\delta = x$ .

- a.  $\frac{32}{3}$  **Correct Choice**
- b.  $\frac{32}{5}$
- c.  $\frac{16}{3}$
- d.  $\frac{16}{5}$
- e.  $\frac{8}{3}$



**Solution:**  $M = \iint \delta dA = \int_0^2 \int_{-2x}^{2x} x dy dx = \int_0^2 [xy]_{y=-2x}^{2x} dx = \int_0^2 4x^2 dx = \frac{4x^3}{3} \Big|_0^2 = \frac{32}{3}$

5. Find the center of mass of a triangular plate with vertices  $(0,0)$ ,  $(2,4)$  and  $(2,-4)$  whose surface mass density is  $\delta = x$ .

- a.  $(\frac{2}{3}, 0)$
- b.  $(0, \frac{2}{3})$
- c.  $(\frac{3}{2}, 0)$  **Correct Choice**
- d.  $(0, \frac{3}{2})$
- e.  $(16, 0)$

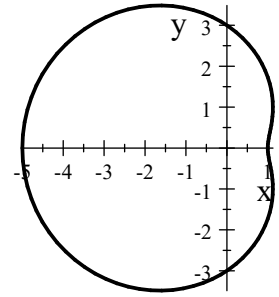
**Solution:**  $\bar{y} = 0$  by symmetry.

$$M_x = \iint x \delta dA = \int_0^2 \int_{-2x}^{2x} x^2 dy dx = \int_0^2 [x^2 y]_{y=-2x}^{2x} dx = \int_0^2 4x^3 dx = x^4 \Big|_0^2 = 16 \quad \bar{x} = \frac{M_x}{M} = \frac{16 \cdot 3}{32} = \frac{3}{2}$$

6. Find the area inside the limaçon

$$r = 3 - 2 \cos \theta.$$

- a.  $\pi$
- b.  $3\pi$
- c.  $7\pi$
- d.  $9\pi$
- e.  $11\pi$     Correct Choice



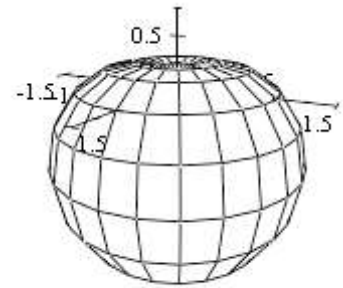
**Solution:** 
$$A = \iint 1 \, dA = \int_0^{2\pi} \int_0^{3-2\cos\theta} r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r^2}{2} \right]_{r=0}^{3-2\cos\theta} d\theta = \frac{1}{2} \int_0^{2\pi} (3 - 2 \cos \theta)^2 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (9 - 12 \cos \theta + 4 \cos^2 \theta) d\theta = \frac{1}{2} \int_0^{2\pi} (9 - 12 \cos \theta + 2(1 + \cos 2\theta)) d\theta$$

$$= \frac{1}{2} [11\theta - 12 \sin \theta + \sin 2\theta]_0^{2\pi} = 11\pi$$

7. Find the volume of the apple whose surface is given in spherical coordinates by  $\rho = 1 - \cos \varphi$ .

- a.  $2\pi$
- b.  $4\pi$
- c.  $\frac{2\pi}{3}$
- d.  $\frac{8\pi}{3}$     Correct Choice
- e.  $\frac{16\pi}{3}$



**Solution:** In spherical coordinates,  $dV = \rho^2 \sin \varphi$ .

$$V = \iiint 1 \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \int_0^{\pi} \left[ \frac{\rho^3}{3} \right]_{\rho=0}^{1-\cos\varphi} \sin \varphi \, d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\pi} (1 - \cos \varphi)^3 \sin \varphi \, d\varphi = \frac{2\pi}{3} \left[ \frac{(1 - \cos \varphi)^4}{4} \right]_0^{\pi} = \frac{2\pi}{3} \frac{2^4}{4} = \frac{8\pi}{3}$$

8. Find the normal to the parametric surface  $\vec{R}(u, v) = (u + v, u - v, uv)$

- a.  $(u + v, v - u, -2)$     Correct Choice
- b.  $(u + v, u - v, -2)$
- c.  $(u + v, v - u, 2)$
- d.  $(u + v, u - v, 2)$
- e.  $(2, 0, u + v)$

**Solution:** The tangent vectors and normal vector are:

$$\vec{e}_u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} \quad \vec{e}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & u \\ 1 & -1 & v \end{vmatrix}$$

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{i}(u + v) - \hat{j}(u - v) + \hat{k}(-1 - 1) = (u + v, v - u, -2)$$

9. Find the mass of the half cylinder **surface**  $x^2 + y^2 = 4$  for  $y \geq 0$  and  $0 \leq z \leq 3$ , with surface mass density  $\delta = y$ . Parametrize the surface as

$$\vec{R}(\theta, z) = (2 \cos \theta, 2 \sin \theta, z)$$

- a. 0
- b. 24    Correct Choice
- c. 48
- d. 72
- e. 96

**Solution:** The tangent vectors and normal vector are:

$$\vec{e}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 \sin \theta & 2 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{i}(2 \cos \theta) - \hat{j}(-2 \sin \theta) + \hat{k}(0) = (2 \cos \theta, 2 \sin \theta, 0)$$

The length of the normal is  $|\vec{N}| = \sqrt{4 \cos^2 \theta + 4 \sin^2 \theta} = 2$ . On the surface,  $\delta = y = 2 \sin \theta$ .

So the mass is:

$$\iint \delta \, dS = \iint y |\vec{N}| \, d\theta \, dz = \int_0^3 \int_0^\pi 4 \sin \theta \, d\theta \, dz = 4[z]_0^3 [-\cos \theta]_0^\pi = 4(3)(-(-1) - (-1)) = 24$$

10. Compute the flux  $\iint \vec{F} \cdot d\vec{S}$  of the vector field  $\vec{F} = (xy, y^2, yz)$  through the half cylinder  $x^2 + y^2 = 4$  for  $y \geq 0$  and  $0 \leq z \leq 3$  oriented toward the positive  $y$ -axis.

- a. 0
- b. 24
- c. 48    Correct Choice
- d. 72
- e. 96

**Solution:** From the previous problem, the normal vector is  $\vec{N} = (2 \cos \theta, 2 \sin \theta, 0)$ .

The vector field on the surface is:

$$\vec{F} = (xy, y^2, yz) = (4 \cos \theta \sin \theta, 4 \sin^2 \theta, 2z \sin \theta)$$

and its dot product with the normal is:

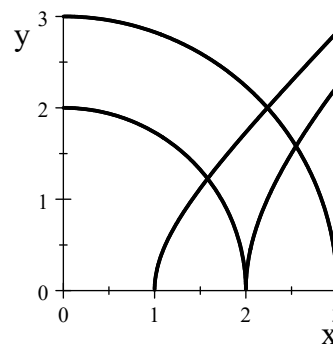
$$\vec{F} \cdot \vec{N} = 8 \cos^2 \theta \sin \theta + 8 \sin^3 \theta + 0 = 8 \sin \theta (\cos^2 \theta + \sin^2 \theta) = 8 \sin \theta$$

So the flux is:

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} \, d\theta \, dz = \int_0^3 \int_0^\pi 8 \sin \theta \, d\theta \, dz = 8[z]_0^3 [-\cos \theta]_0^\pi = 8(3)(-(-1) - (-1)) = 48$$

Work Out: (15 points each. Part credit possible. Show all work.)

11. (15 points) Compute the integral  $\iint xy dA$  over the diamond shaped region in the 1<sup>st</sup> quadrant bounded by  $x^2 + y^2 = 4$ ,  $x^2 + y^2 = 9$ ,  $x^2 - y^2 = 1$ , and  $x^2 - y^2 = 4$ .



**Solution:** Let  $u = x^2 + y^2$  and  $v = x^2 - y^2$ . Then  $4 \leq u \leq 9$  and  $1 \leq v \leq 4$ .  
Add these:  $u + v = 2x^2$ . Subtract these:  $u - v = 2y^2$ .

So the coordinate system is  $(x,y) = \vec{R}(u,v) = \left( \frac{\sqrt{u+v}}{\sqrt{2}}, \frac{\sqrt{u-v}}{\sqrt{2}} \right)$ . And the Jacobian is:

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{u+v}} & \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{u-v}} \\ \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{u+v}} & \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{u-v}} \end{vmatrix} = \left| \frac{1}{8} \frac{-1}{\sqrt{u^2 - v^2}} - \frac{1}{8} \frac{1}{\sqrt{u^2 - v^2}} \right| = \frac{1}{4\sqrt{u^2 - v^2}}$$

The integrand is  $xy = \frac{\sqrt{u+v}}{\sqrt{2}} \frac{\sqrt{u-v}}{\sqrt{2}} = \frac{1}{2} \sqrt{u^2 - v^2}$ . So the integral is:

$$\iint xy dA = \int_1^4 \int_4^9 \frac{1}{2} \sqrt{u^2 - v^2} \frac{1}{4\sqrt{u^2 - v^2}} du dv = \int_1^4 \int_4^9 \frac{1}{8} du dv = \frac{1}{8} (4-1)(9-4) = \frac{15}{8}$$

12. (15 points) Find the mass of the ice cream cone below the paraboloid  $z = 8 - x^2 - y^2$  above the cone  $z = 2\sqrt{x^2 + y^2}$  if the density is  $\delta = \sqrt{x^2 + y^2}$ .



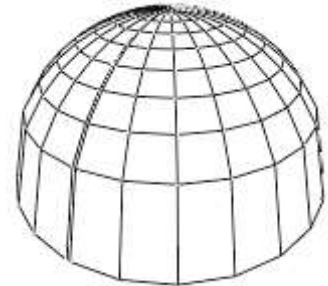
**Solution:** In cylindrical coordinates,  $dV = r dr d\theta dz$ . The density is  $\delta = r$ . The boundaries are  $z = 8 - r^2$  and  $z = 2r$ . We find the intersection by setting these equal:

$$8 - r^2 = 2r \quad r^2 + 2r - 8 = 0 \quad (r - 2)(r + 4) = 0 \quad r = 2$$

$$\begin{aligned} M &= \iint \delta dV = \int_0^{2\pi} \int_0^2 \int_{2r}^{8-r^2} r r dz dr d\theta = 2\pi \int_0^2 [z]_{z=2r}^{8-r^2} r^2 dr = 2\pi \int_0^2 (8 - r^2 - 2r)r^2 dr \\ &= 2\pi \int_0^2 (8r^2 - r^4 - 2r^3) dr = 2\pi \left[ \frac{8r^3}{3} - \frac{r^5}{5} - \frac{2r^4}{4} \right]_0^2 = 2\pi \left( \frac{64}{3} - \frac{32}{5} - 8 \right) = \frac{208}{15}\pi \end{aligned}$$

13. (15 points) Find the centroid of the hemisphere

$$0 \leq z \leq \sqrt{9 - x^2 - y^2}.$$



**Solution:** We know  $\bar{x} = \bar{y} = 0$  by symmetry. In spherical coordinates,  $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$ . The volume is:

$$V = \iiint 1 dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^2 \sin \varphi d\rho d\varphi d\theta = \frac{1}{2} \frac{4}{3} \pi (3)^3 = 18\pi$$

Here, we set up the integral but then used the high school formula. The  $z$ -moment is:

$$\begin{aligned} V_z &= \iiint z dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho \cos \varphi) \rho^2 \sin \varphi d\rho d\varphi d\theta \\ &= \int_0^{2\pi} 1 d\theta \int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi \int_0^3 \rho^3 d\rho = 2\pi \left[ -\frac{\cos^2 \varphi}{2} \right]_0^{\pi/2} \left[ \frac{\rho^4}{4} \right]_0^3 \\ &= 2\pi \left( 0 - -\frac{1}{2} \right) \left( \frac{3^4}{4} \right) = \frac{81\pi}{4} \end{aligned}$$

So the  $z$ -component of the centroid is:

$$\bar{z} = \frac{V_z}{V} = \frac{81\pi}{4 \cdot 18\pi} = \frac{9}{8}$$

So the centroid is  $(\bar{x}, \bar{y}, \bar{z}) = \left( 0, 0, \frac{9}{8} \right)$ .