Name________________________

1-10	/60	15	/15
14	/15	16	/15
		Total	/105

MATH 251 Exam 3 Version H Fall 2017

Sections 515 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Compute $\int_0^2 \int_{x^2}^{2x} 2y\,dy\,dx$.

a. $-\frac{56}{15}$

b. $-\frac{32}{15}$

c. $\frac{32}{15}$

d. $\frac{64}{15}$ Correct Choice

e. $\frac{128}{15}$

Solution: $\int_0^2 \int_{x^2}^{2x} 2y\,dy\,dx = \int_0^2 \Big[y^2\Big]_{y=x^2}^{2x} dx = \int_0^2 (4x^2 - x^4)\,dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$

2. Find the average value of the function $f(x,y) = x^2 \sin y$ on the rectangle $[0,3] \times [0,\pi]$.

a. $\frac{18}{\pi}$

b. $\frac{6}{\pi}$ Correct Choice

c. 3π

d. 18

e. 54π

Solution: The area is $A = 3\pi$. The integral of f is

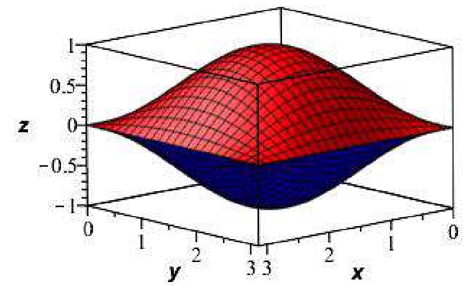
$$\iint f\,dA = \int_0^\pi \int_0^3 x^2 \sin y\,dx\,dy = \left[\frac{x^3}{3}\right]_0^3 \Big[-\cos y\Big]_0^\pi = 9(--1 --1) = 18$$

So the average is $f_{\text{ave}} = \frac{1}{A} \iint f\,dA = \frac{18}{3\pi} = \frac{6}{\pi}$

3. Find the volume of the ravioli bounded by

$$-\sin x \sin y \le z \le \sin x \sin y$$

for $0 \le x \le \pi$ and $0 \le y \le \pi$.



a. 2

b. 4

c. 8 Correct Choice

d. π^2

e. $4\pi^2$

Solution: $A = \int_0^\pi \int_0^\pi (\sin x \sin y - -\sin x \sin y)\,dx\,dy = 2\int_0^\pi \int_0^\pi \sin x \sin y\,dx\,dy$

$$= 2\int_0^\pi \sin x\,dx \int_0^\pi \sin y\,dy = 2[-\cos x]_0^\pi[-\cos y]_0^\pi = 2(2)(2) = 8$$

4. Find the mass of a triangular plate with vertices $(0,0)$, $(2,4)$ and $(2,-4)$ whose surface mass density is $\delta = x$.

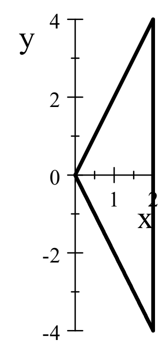
a. $\frac{32}{3}$ Correct Choice

b. $\frac{32}{5}$

c. $\frac{16}{3}$

d. $\frac{16}{5}$

e. $\frac{8}{3}$



Solution: $M = \iint \delta\,dA = \int_0^2 \int_{-2x}^{2x} x\,dy\,dx = \int_0^2 \Big[xy\Big]_{y=-2x}^{2x} dx = \int_0^2 4x^2\,dx = \frac{4x^3}{3}\Big|_0^2 = \frac{32}{3}$

5. Find the center of mass of a triangular plate with vertices $(0,0)$, $(2,4)$ and $(2,-4)$ whose surface mass density is $\delta = x$.

a. $\left(\frac{2}{3},0\right)$

b. $\left(0,\frac{2}{3}\right)$

c. $\left(\frac{3}{2},0\right)$ Correct Choice

d. $\left(0,\frac{3}{2}\right)$

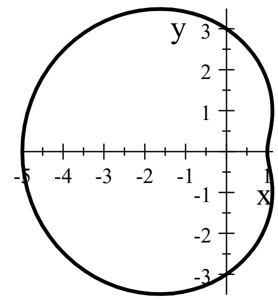
e. $(16,0)$

Solution: $\bar{y} = 0$ by symmetry.

$$M_x = \iint x\delta\,dA = \int_0^2 \int_{-2x}^{2x} x^2\,dy\,dx = \int_0^2 \Big[x^2 y\Big]_{y=-2x}^{2x} dx = \int_0^2 4x^3\,dx = x^4\Big|_0^2 = 16 \qquad \bar{x} = \frac{M_x}{M} = \frac{16 \cdot 3}{32} = \frac{3}{2}$$

6. Find the area inside the limacon

$$r = 3 - 2\cos\theta.$$



a. π

b. 3π

c. 7π

d. 9π

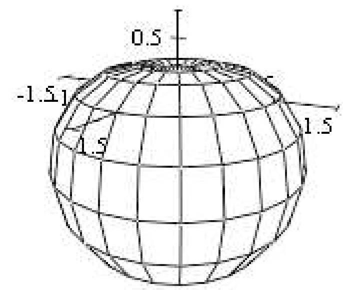
e. 11π Correct Choice

Solution: $A = \iint 1\,dA = \int_0^{2\pi}\int_0^{3-2\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{3-2\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi}(3-2\cos\theta)^2\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}(9 - 12\cos\theta + 4\cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}(9 - 12\cos\theta + 2(1+\cos 2\theta))\,d\theta$

$= \frac{1}{2}[11\theta - 12\sin\theta + \sin 2\theta]_0^{2\pi} = 11\pi$

7. Find the volume of the apple whose surface is given in spherical coordinates by $\rho = 1 - \cos\varphi$.



a. 2π

b. 4π

c. $\frac{2\pi}{3}$

d. $\frac{8\pi}{3}$ Correct Choice

e. $\frac{16\pi}{3}$

Solution: In spherical coordinates, $dV = \rho^2\sin\varphi$.

$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\int_0^{\pi}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{1-\cos\varphi}\sin\varphi\,d\varphi$

$= \frac{2\pi}{3}\int_0^{\pi}(1-\cos\varphi)^3\sin\varphi\,d\varphi = \frac{2\pi}{3}\left[\frac{(1-\cos\varphi)^4}{4}\right]_0^{\pi} = \frac{2\pi}{3}\frac{2^4}{4} = \frac{8\pi}{3}$

8. Find the normal to the parametric surface $\vec{R}(u,v) = (u+v, u-v, uv)$

a. $(u+v, v-u, -2)$ Correct Choice

b. $(u+v, u-v, -2)$

c. $(u+v, v-u, 2)$

d. $(u+v, u-v, 2)$

e. $(2, 0, u+v)$

Solution: The tangent vectors and normal vector are:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (1 & 1 & v) \\ \vec{e}_v = & (1 & -1 & u) \end{array} \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(u+v) - \hat{\jmath}(u-v) + \hat{k}(-1-1) = (u+v, v-u, -2)$$

9. Find the mass of the half cylinder **surface** $x^2 + y^2 = 4$ for $y \ge 0$ and $0 \le z \le 3$, with surface mass density $\delta = y$. Parametrize the surface as

$$\vec{R}(\theta,z) = (2\cos\theta, 2\sin\theta, z)$$

a. 0

b. 24 Correct Choice

c. 48

d. 72

e. 96

Solution: The tangent vectors and normal vector are:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-2\sin\theta & 2\cos\theta & 0) \\ \vec{e}_z = & (\ 0 & 0 & 1) \end{array} \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(2\cos\theta) - \hat{\jmath}(-2\sin\theta) + \hat{k}(0) = (2\cos\theta, 2\sin\theta, 0)$$

The length of the normal is $\left|\vec{N}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$. On the surface, $\delta = y = 2\sin\theta$.
So the mass is:

$$\iint \delta\, dS = \iint y \left|\vec{N}\right| d\theta\, dz = \int_0^3 \int_0^\pi 4\sin\theta\, d\theta\, dz = 4\big[z\big]_0^3 \big[-\cos\theta\big]_0^\pi = 4(3)(--1--1) = 24$$

10. Compute the flux $\iint \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = (xy, y^2, yz)$ through the half cylinder $x^2 + y^2 = 4$ for $y \ge 0$ and $0 \le z \le 3$ oriented toward the positive y-axis.

a. 0

b. 24

c. 48 Correct Choice

d. 72

e. 96

Solution: From the previous problem, the normal vector is $\vec{N} = (2\cos\theta, 2\sin\theta, 0)$.
The vector field on the surface is:

$$\vec{F} = (xy, y^2, yz) = (4\cos\theta\sin\theta, 4\sin^2\theta, 2z\sin\theta)$$

and its dot product with the normal is:

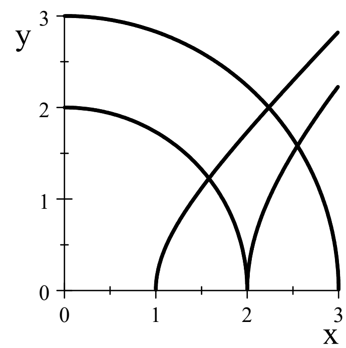
$$\vec{F} \cdot \vec{N} = 8\cos^2\theta\sin\theta + 8\sin^3\theta + 0 = 8\sin\theta(\cos^2\theta + \sin^2\theta) = 8\sin\theta$$

So the flux is:

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N}\, d\theta\, dz = \int_0^3 \int_0^\pi 8\sin\theta\, d\theta\, dz = 8\big[z\big]_0^3 \big[-\cos\theta\big]_0^\pi = 8(3)(--1--1) = 48$$

Work Out: (15 points each. Part credit possible. Show all work.)

11. (15 points) Compute the integral $\iint xy\,dA$ over the diamond shaped region in the 1st quadrant bounded by $x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$, and $x^2-y^2=4$.



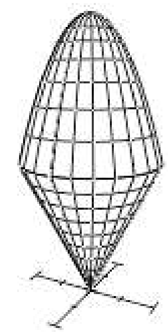
Solution: Let $u=x^2+y^2$ and $v=x^2-y^2$. Then $4\le u\le 9$ and $1\le v\le 4$.
Add these: $u+v=2x^2$. Subtract these: $u-v=2y^2$.
So the coordinate system is $(x,y)=\vec{R}(u,v)=\left(\frac{\sqrt{u+v}}{\sqrt{2}},\frac{\sqrt{u-v}}{\sqrt{2}}\right)$. And the Jacobian is:

$$J=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\begin{matrix}\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{u+v}} & \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{u-v}}\\ \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{u+v}} & \frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{u-v}}\end{matrix}\right|=\left|\frac{1}{8}\frac{-1}{\sqrt{u^2-v^2}}-\frac{1}{8}\frac{1}{\sqrt{u^2-v^2}}\right|=\frac{1}{4\sqrt{u^2-v^2}}$$

The integrand is $xy=\frac{\sqrt{u+v}}{\sqrt{2}}\frac{\sqrt{u-v}}{\sqrt{2}}=\frac{1}{2}\sqrt{u^2-v^2}$. So the integral is:

$$\iint xy\,dA=\int_1^4\int_4^9\frac{1}{2}\sqrt{u^2-v^2}\,\frac{1}{4\sqrt{u^2-v^2}}\,du\,dv=\int_1^4\int_4^9\frac{1}{8}\,du\,dv=\frac{1}{8}(4-1)(9-4)=\frac{15}{8}$$

12. (15 points) Find the mass of the ice cream cone below the paraboloid $z = 8 - x^2 - y^2$ above the cone $z = 2\sqrt{x^2 + y^2}$ if the density is $\delta = \sqrt{x^2 + y^2}$.



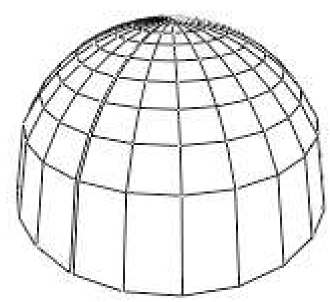
Solution: In cylindrical coordinates, $dV = r\,dr\,d\theta\,dz$. The density is $\delta = r$.
The boundaries are $z = 8 - r^2$ and $z = 2r$. We find the intersection by setting these equal:

$$8 - r^2 = 2r \qquad r^2 + 2r - 8 = 0 \qquad (r-2)(r+4) = 0 \qquad r = 2$$

$$M = \iint \delta\,dV = \int_0^{2\pi}\int_0^2\int_{2r}^{8-r^2} r\,r\,dz\,dr\,d\theta = 2\pi\int_0^2 \Big[z\Big]_{z=2r}^{8-r^2} r^2\,dr = 2\pi\int_0^2 (8 - r^2 - 2r)r^2\,dr$$

$$= 2\pi\int_0^2 (8r^2 - r^4 - 2r^3)\,dr = 2\pi\left[\frac{8r^3}{3} - \frac{r^5}{5} - \frac{2r^4}{4}\right]_0^2 = 2\pi\left(\frac{64}{3} - \frac{32}{5} - 8\right) = \frac{208}{15}\pi$$

13. (15 points) Find the centroid of the hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$.



Solution: We know $\bar{x} = \bar{y} = 0$ by symmetry. In spherical coordinates, $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$.
The volume is:

$$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{1}{2}\,\frac{4}{3}\pi(3)^3 = 18\pi$$

Here, we set up the integral but then used the high school formula. The z-moment is:

$$V_z = \iiint z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 (\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta \int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\int_0^3 \rho^3\,d\rho = 2\pi\left[-\frac{\cos^2\varphi}{2}\right]_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_0^3$$

$$= 2\pi\left(0 - -\frac{1}{2}\right)\left(\frac{3^4}{4}\right) = \frac{81\pi}{4}$$

So the z-component of the centroid is:

$$\bar{z} = \frac{V_z}{V} = \frac{81\pi}{4 \cdot 18\pi} = \frac{9}{8}$$

So the centriod is $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \frac{9}{8}\right)$.