Name________________________

MATH 251 Final Exam Version A Fall 2017

Sections 515 Solutions P. Yasskin

1-10	/50	13	/15
11	/ 5	14	/15
12	/20	Total	/105

Multiple Choice: (5 points each. No part credit.)

1. A wire has the shape of the helix curve $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 4\theta)$ for $0 \le \theta \le \pi$ and has linear density $\delta = 2y$. Find the total mass of the wire.

a. 80

b. 60 Correct Choice

c. 40

d. 20

e. 10

Solution: $\delta = 2y = 6\sin\theta$

$\vec{v} = (-3\sin\theta, 3\cos\theta, 4)$ $|\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta + 16} = 5$ $ds = |\vec{v}|\,d\theta = 5\,d\theta$

$M = \int \delta\,ds = \int_0^\pi 2y|\vec{v}|\,d\theta = \int_0^\pi 6\sin\theta\, 5\,d\theta = 30 - \cos\theta\Big|_0^\pi = 60$

2. A wire has the shape of the helix curve $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 4\theta)$ for $0 \le \theta \le \pi$ and has linear density $\delta = 2y$. Find the y-component of the center of mass of the wire.

a. $\dfrac{3\pi}{4}$ Correct Choice

b. $\dfrac{4}{3\pi}$

c. 45π

d. $\dfrac{1}{45\pi}$

e. 3π

Solution: From the previous problem, $M = 60$.

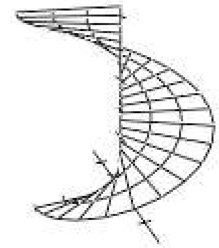
$M_y = \int y\,\delta\,ds = \int_0^\pi y 2y|\vec{v}|\,d\theta = \int_0^\pi 18\sin^2\theta\, 5\,d\theta = 45\int_0^\pi (1 - \cos 2\theta)\,d\theta = 45\left[\theta - \dfrac{\sin 2\theta}{2}\right]_0^\pi = 45\pi$

So the y-component of the center of mass is $\bar{y} = \dfrac{M_y}{M} = \dfrac{45\pi}{60} = \dfrac{3\pi}{4}$

3. The spiral ramp shown at the right may be parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 3$ and $0 \le \theta \le 2\pi$.



Find the total mass, if the surface density is

$$\delta = \sqrt{x^2+y^2}$$

a. $\frac{2\pi}{3}(5^{3/2}-1)$

b. $\frac{2\pi}{3}5^{3/2}$

c. $\frac{2\pi}{3}(10^{3/2}-1)$ Correct Choice

d. $\frac{2\pi}{3}10^{3/2}$

e. $\frac{2\pi}{3}(10^{3/2}-5^{3/2})$

Solution: The tangent vectors, normal vector and its length are:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta & 0) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 1) \end{vmatrix} \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(\sin\theta) - \hat{\jmath}(\cos\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (\sin\theta, -\cos\theta, r)$$

$$|\vec{N}| = \sqrt{\sin^2\theta+\cos^2\theta+r^2} = \sqrt{1+r^2}$$

The density is $\delta = \sqrt{x^2+y^2} = r$. So the mass is

$$M = \iint \delta\, dS = \iint \delta\,|\vec{N}|\, dr\,d\theta = \int_0^{2\pi}\int_0^3 r\sqrt{1+r^2}\, dr\,d\theta = 2\pi\left[\frac{(1+r^2)^{3/2}}{3}\right]_0^3 = \frac{2\pi}{3}(10^{3/2}-1)$$

4. Consider the spiral ramp described in the previous problem.
Find the flux of the vector field $\vec{F} = (0,0,z)$ **upward** through the spiral ramp.

a. $-9\pi^2$

b. $-4\pi^2$

c. 0

d. $4\pi^2$

e. $9\pi^2$ Correct Choice

Solution: From the previous problem $\vec{N} = (\sin\theta, -\cos\theta, r)$ which is oriented upward.
On the ramp, $\vec{F} = (0,0,\theta)$. So the flux is

$$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \theta\, r\, dr\,d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3 = 9\pi^2$$

5. Compute $\int_A^B \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x + y, x + 2y)$ along the line segment from $A = (2,2)$ to $B = (3,3)$.
Hint: Find a scalar potential.

a. 15 Correct Choice
b. 6
c. 0
d. -6
e. -15

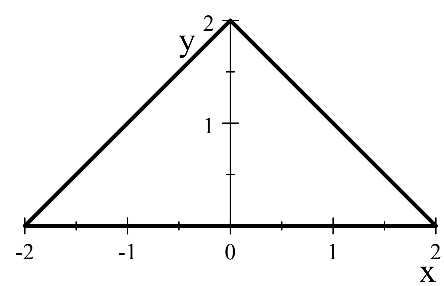
Solution: $\vec{F} = (2x + y, x + 2y) = \vec{\nabla} f$ for $f = x^2 + xy + y^2$ since $\partial_x f = 2x + y$ and $\partial_y f = x + 2y$.
By the F.T.C.C. $\int \vec{F} \cdot d\vec{s} = \int_{(2,3)}^{(3,3)} \vec{\nabla} f \cdot d\vec{s} = f(3,3) - f(2,2) = (9 + 9 + 9) - (4 + 4 + 4) = 15$

6. Compute $\int_A^B \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-y, x, 2)$ along the helix $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 3\theta)$
from $A = (4,0,0)$ to $B = (4,0,6\pi)$.

a. 0
b. 40π
c. 42π
d. 44π Correct Choice
e. 46π

Solution: Since $\vec{\nabla} \times \vec{F} = (0,0,2) \neq \vec{0}$, there is no scalar potential and we cannot use the F.T.C.C.
So we compute the line integral from its definition:
$\vec{v} = (-4\sin\theta, 4\cos\theta, 3)$ $\quad \vec{F}\big|_{\vec{r}(\theta)} = (-4\sin\theta, 4\cos\theta, 2)$ $\quad \vec{F} \cdot \vec{v} = 16\sin^2\theta + 16\cos^2\theta + 6 = 22$
$\int_A^B \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 22\, d\theta = 44\pi$

7. Compute $\oint_{\partial T} (\sin x + 5y)\, dx + (3x + \cos y)\, dy$
clockwise around the complete boundary of the triangle shown at the right.
Hint: Use a Theorem.

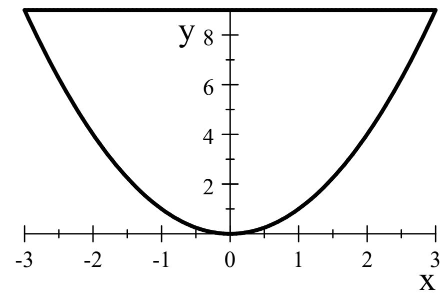


a. 12
b. 8 Correct Choice
c. 0
d. -8
e. -12

Solution: $P = \sin x + 5y$ $\quad Q = 3x + \cos y$ $\quad \partial_x Q - \partial_y P = 3 - 5 = -2$
By Green's Theorem, (There is a minus sign, since we want clockwise.)

$$\oint_{\partial T} \vec{F} \cdot d\vec{s} = -\iint_T (\partial_x Q - \partial_y P)\, dx\, dy = -\iint_T -2\, dx\, dy = 2\big(\text{area}\big) = 2(4) = 8$$

8. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (x^2 y, 2x^3)$ along the piece of the parabola $y = x^2$ from $(-3,9)$ to $(3,9)$ followed by the line segment from $(3,9)$ back to $(-3,9)$.
Hint: Use Green's Theorem.



a. 0
b. 81
c. 162
d. 324 Correct Choice
e. 405

Solution: $\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy$ with $P = x^2 y$ and $Q = 2x^3$. By Green's Theorem,
$\oint \vec{F} \cdot d\vec{s} = \int_{-3}^{3} \int_{x^2}^{9} (\partial_x Q - \partial_y P)\,dy\,dx = \int_{-3}^{3} \int_{x^2}^{9} (6x^2 - x^2)\,dy\,dx = \int_{-3}^{3} 5x^2 \Big[y \Big]_{x^2}^{9} dx$
$= \int_{-3}^{3} (45x^2 - 5x^4)\,dx = [15x^3 - x^5]_{-3}^{3} = 2(15 \cdot 3^3 - 3^5) = 324$

9. Compute $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S}$ over the complete surface of the cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 3$ oriented out from the cylinder for $\vec{F} = (xz, yz, z^2)$.
Hint: Use Gauss' Theorem.

a. 24π
b. 36π
c. 72π Correct Choice
d. 144π
e. 288π

Solution: By Gauss' Theorem, $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S} = \iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\,dV$ and $\vec{\nabla} \cdot \vec{F} = z + z + 2z = 4z$
$\iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\,dV = \int_0^3 \int_0^{2\pi} \int_0^2 4zr\,dr\,d\theta\,dz = 4\left[\frac{z^2}{2}\right]_0^3 (2\pi) \left[\frac{r^2}{2}\right]_0^2 = 4\left(\frac{9}{2}\right)(2\pi)(2) = 72\pi$

10. Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx$ in problem (11).
Select its value here:

a. $\frac{1}{2}\sin 16$

b. $\frac{1}{4}\sin 16$ Correct Choice

c. $\frac{1}{2}\sin 4$

d. $\frac{1}{4}\sin 4$

e. $\frac{1}{4}\sin 2$

Solution: Reverse the order of integration:

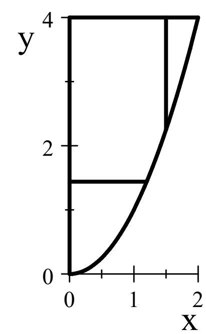
$$\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^2}{2}\cos(y^2) \right]_{x=0}^{\sqrt{y}} dy = \int_0^4 \frac{y}{2}\cos(y^2)\,dy$$

Substitute: $u = y^2$ $\quad du = 2y\,dy$ $\quad y\,dy = \frac{1}{2}du$

$$\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u)$$

$$= \frac{1}{4}\sin(y^2)\Big|_0^4 = \frac{1}{4}\sin 16$$



Work Out: (Points indicated. Part credit possible. Show all work.)

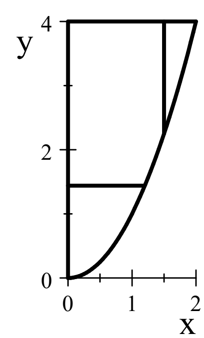
11. (5 points) Sketch the region of integration

for the integral $\int_0^2 \int_{x^2}^4 x\cos(y^2)$.

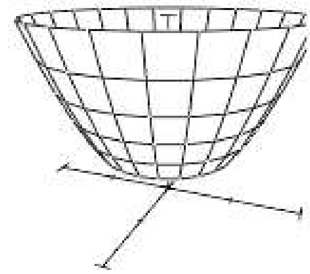
Shade in the region.

Compute its value in problem (10).

Solution:



12. (20 points) Verify Stokes' Theorem $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (-2yz, 2xz, z^2)$ and the **surface** which is the piece of the paraboloid P given by $z = x^2 + y^2$ between $z = 1$ and $z = 4$ oriented up and in.
Notice that the boundary of P is two circles.

Be sure to check orientations. Use the following steps:

a. The paraboloid may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ for $1 \le r \le 2$.

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\cos\theta, & \sin\theta, & 2r) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0\ ) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ has the correct orientation.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2yz & 2xz & z^2 \end{vmatrix} = \hat{\imath}(-2x) - \hat{\jmath}(2y) + \hat{k}(2z + 2z) = (-2x, -2y, 4z)$$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (-2r\cos\theta, -2r\sin\theta, 4r^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 4r^3\cos^2\theta + 4r^3\sin^2\theta + 4r^3 = 8r^3$$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_P \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_1^2 8r^3\, dr\, d\theta = 2\pi[2r^4]_{r=1}^2 = 4\pi(16 - 1) = 60\pi$$

Recall $\vec{F} = (-2yz, 2xz, z^2)$

b. Parametrize the upper circle U and compute the line integral.

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$ oriented correctly counterclockwise

$\vec{F}\big|_{\vec{r}(\theta)} = (-16\sin\theta, 16\cos\theta, 16)$

$$\oint_U \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 32\sin^2\theta + 32\cos^2\theta\, d\theta = \int_0^{2\pi} 32\, d\theta = 64\pi$$

c. Parametrize the lower circle L and compute the line integral.

$\vec{r}(\theta) = (\cos\theta, \sin\theta, 1)$

$\vec{v}(\theta) = (-\sin\theta, \cos\theta, 0)$ oriented counterclockwise, need clockwise

Rev $\vec{v}(\theta) = (\sin\theta, -\cos\theta, 0)$

$\vec{F}\big|_{\vec{r}(\theta)} = (-2\sin\theta, 2\cos\theta, 1)$

$$\oint_L \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -2\sin^2\theta - 2\cos^2\theta\, d\theta = -\int_0^{2\pi} 2\, d\theta = -4\pi$$

d. Combine $\oint_U \vec{F} \cdot d\vec{s}$ and $\oint_L \vec{F} \cdot d\vec{s}$ to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$

$$\oint_{\partial P} \vec{F} \cdot d\vec{s} = \oint_U \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 64\pi - 4\pi = 60\pi$$

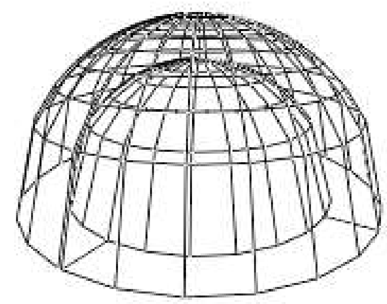
which agrees with part (a).

13. (15 points) (Also replaces Exam 3 #12.)

Find the mass of the solid between the hemispheres

$z = \sqrt{9 - x^2 - y^2}$ and $z = \sqrt{16 - x^2 - y^2}$

for $z \ge 0$ if the density is $\delta = \dfrac{1}{x^2 + y^2 + z^2}$.



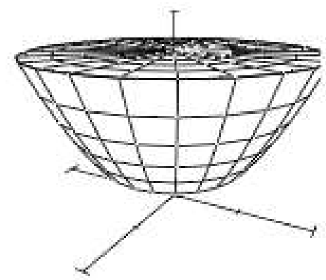
Solution: In spherical coordinates, $3 \le \rho \le 4$ and $\delta = \dfrac{1}{\rho^2}$ and $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

$$M = \iiint \delta \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_3^4 \frac{1}{\rho^2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \Big[\rho \Big]_3^4 \Big[-\cos\varphi \Big]_0^{\pi/2}$$
$$= 2\pi(4-3)(-0 - -1) = 2\pi$$

14. (15 points) (Also replaces Exam 3 #13.)

Find the **centroid** of the **solid** inside the paraboloid $z = x^2 + y^2$ for $1 \le z \le 4$.

Hint: Put the differentials in the order $dr \, dz \, d\theta$.



Solution: In cylindrical coordinates, $dV = r \, dr \, d\theta \, dz$ and the paraboloid is $z = r^2$ or $r = \sqrt{z}$.

The volume is:

$$V = \iiint 1 \, dV = \int_0^{2\pi} \int_1^4 \int_0^{\sqrt{z}} r \, dr \, dz \, d\theta = 2\pi \int_1^4 \left[\frac{r^2}{2} \right]_0^{\sqrt{z}} dz = \pi \int_1^4 z \, dz = \pi \left[\frac{z^2}{2} \right]_1^4$$
$$= \frac{\pi}{2}(16 - 1) = \frac{15\pi}{2}$$

The z-moment is:

$$V_z = \iiint z \, dV = \int_0^{2\pi} \int_1^4 \int_0^{\sqrt{z}} z \, r \, dr \, dz \, d\theta = 2\pi \int_1^4 z \left[\frac{r^2}{2} \right]_0^{\sqrt{z}} dz$$
$$= \pi \int_1^4 z^2 \, dz = \pi \left[\frac{z^3}{3} \right]_1^4 = \frac{\pi}{3}(64 - 1) = 21\pi$$

So the z-component of the centroid is:

$$\bar{z} = \frac{V_z}{V} = 21\pi \frac{2}{15\pi} = \frac{14}{5} = 2.8$$

We know $\bar{x} = \bar{y} = 0$ by symmetry. So the centriod is $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{5}\right)$.