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MATH 251 Final Exam Version H Fall 2017

Sections 200 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1-10	/50	13	/15
11	/ 5	14	/15
12	/20	Total	/105

1. A wire has the shape of the helix curve $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 3\theta)$ for $0 \le \theta \le \pi$ and has linear density $\delta = 2y$. Find the total mass of the wire.

a. 80 Correct Choice

b. 60

c. 40

d. 20

e. 10

Solution: $\delta = 2y = 8\sin\theta$

$\vec{v} = (-4\sin\theta, 4\cos\theta, 3)$ $|\vec{v}| = \sqrt{16\sin^2\theta + 16\cos^2\theta + 9} = 5$ $ds = |\vec{v}|\,d\theta = 5\,d\theta$

$M = \int \delta\,ds = \int_0^\pi 2y|\vec{v}|\,d\theta = \int_0^\pi 8\sin\theta\, 5\,d\theta = 40 - \cos\theta\Big|_0^\pi = 80$

2. A wire has the shape of the helix curve $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 3\theta)$ for $0 \le \theta \le \pi$ and has linear density $\delta = 2y$. Find the y-component of the center of mass of the wire.

a. 80π

b. $\dfrac{1}{80\pi}$

c. 40π

d. $\dfrac{1}{40\pi}$

e. π Correct Choice

Solution: From the previous problem, $M = 80$.

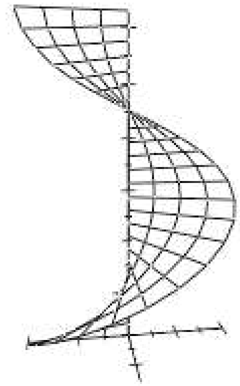
$M_y = \int y\,\delta\,ds = \int_0^\pi y 2y|\vec{v}|\,d\theta = \int_0^\pi 32\sin^2\theta\, 5\,d\theta = 80\int_0^\pi (1 - \cos 2\theta)\,d\theta = 80\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi = 80\pi$

So the y-component of the center of mass is $\bar{y} = \dfrac{M_y}{M} = \dfrac{80\pi}{80} = \pi$

3. The spiral ramp shown at the right may be parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.



Find the total mass, if the surface density is

$$\delta = \sqrt{x^2+y^2}$$

a. $\frac{2\pi}{3}(5^{3/2}-1)$ Correct Choice

b. $\frac{2\pi}{3}5^{3/2}$

c. $\frac{2\pi}{3}(10^{3/2}-1)$

d. $\frac{2\pi}{3}10^{3/2}$

e. $\frac{2\pi}{3}(10^{3/2}-5^{3/2})$

Solution: The tangent vectors, normal vector and its length are:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta & 0) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 1) \end{vmatrix} \qquad \begin{aligned} \vec{N} &= \vec{e}_u \times \vec{e}_v = \hat{\imath}(\sin\theta) - \hat{\jmath}(\cos\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) \\ &= (\sin\theta, -\cos\theta, r) \end{aligned}$$

$$\left|\vec{N}\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}$$

The density is $\delta = \sqrt{x^2+y^2} = r$. So the mass is

$$M = \iint \delta\, dS = \iint \delta \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{1+r^2}\, dr\, d\theta = 2\pi\left[\frac{(1+r^2)^{3/2}}{3}\right]_0^2 = \frac{2\pi}{3}(5^{3/2}-1)$$

4. Consider the spiral ramp described in the previous problem.
Find the flux of the vector field $\vec{F} = (0,0,z)$ **upward** through the spiral ramp.

a. $-9\pi^2$

b. $-4\pi^2$

c. 0

d. $4\pi^2$ Correct Choice

e. $9\pi^2$

Solution: From the previous problem $\vec{N} = (\sin\theta, -\cos\theta, r)$ which is oriented upward.
On the ramp, $\vec{F} = (0,0,\theta)$. So the flux is

$$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \theta r\, dr\, d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 = 4\pi^2$$

5. Compute $\int_A^B \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x+y, x+2y)$ along the line segment from $A = (2,1)$ to $B = (1,3)$.
Hint: Find a scalar potential.

a. 15
b. 6 Correct Choice
c. 0
d. -6
e. -15

Solution: $\vec{F} = (2x+y, x+2y) = \vec{\nabla} f$ for $f = x^2 + xy + y^2$ since $\partial_x f = 2x+y$ and $\partial_y f = x+2y$.
By the F.T.C.C. $\int \vec{F} \cdot d\vec{s} = \int_{(2,1)}^{(1,3)} \vec{\nabla} f \cdot d\vec{s} = f(1,3) - f(2,1) = (1+3+9) - (4+2+1) = 6$

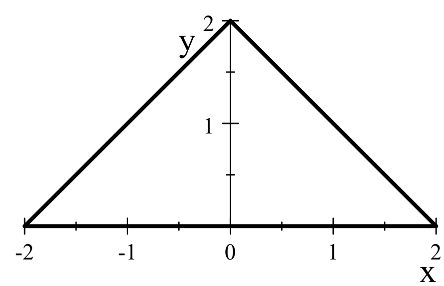
6. Compute $\int_A^B \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-y, x, 3)$ along the helix $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 4\theta)$
from $A = (3,0,0)$ to $B = (3,0,8\pi)$.

a. 0
b. 40π
c. 42π Correct Choice
d. 44π
e. 46π

Solution: Since $\vec{\nabla} \times \vec{F} = (0,0,2) \neq \vec{0}$, there is no scalar potential and we cannot use the F.T.C.C.
So we compute the line integral from its definition:
$\vec{v} = (-3\sin\theta, 3\cos\theta, 4)$ $\quad \vec{F}\big|_{\vec{r}(\theta)} = (-3\sin\theta, 3\cos\theta, 3)$ $\quad \vec{F} \cdot \vec{v} = 9\sin^2\theta + 9\cos^2\theta + 12 = 21$
$\int_A^B \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 21\, d\theta = 42\pi$

7. Compute $\oint_{\partial T} (\sin x + 5y)\, dx + (2x + \cos y)\, dy$

clockwise around the complete boundary of the triangle shown at the right.
Hint: Use a Theorem.

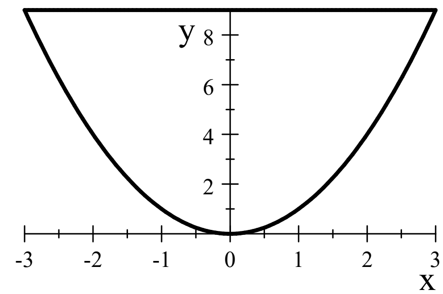


a. 12 Correct Choice
b. 8
c. 0
d. -8
e. -12

Solution: $P = \sin x + 5y$ $\quad Q = 2x + \cos y$ $\quad \partial_x Q - \partial_y P = 2 - 5 = -3$
By Green's Theorem, (There is a minus sign, since we want clockwise.)

$$\oint_{\partial T} \vec{F} \cdot d\vec{s} = -\iint_T (\partial_x Q - \partial_y P)\, dx\, dy = -\iint_T -3\, dx\, dy = 3\big(\text{area}\big) = 3(4) = 12$$

8. Compute $\oint \vec{F} \cdot d\vec{n}$ for $\vec{F} = (2x^3, -x^2 y)$ along the piece of the parabola $y = x^2$ from $(-3,9)$ to $(3,9)$ followed by the line segment from $(3,9)$ back to $(-3,9)$.
Hint: Use the 2D Gauss' Theorem.



a. 405
b. 324 Correct Choice
c. 162
d. 81
e. 0

Solution: By the 2D Gauss' Theorem $\oint \vec{F} \cdot d\vec{n} = \iint \vec{\nabla} \cdot \vec{F}\, dA$ and $\vec{\nabla} \cdot \vec{F} = 6x^2 - x^2 = 5x^2$.

So, $$\oint \vec{F} \cdot d\vec{n} = \int_{-3}^{3} \int_{x^2}^{9} 5x^2\, dy\, dx = \int_{-3}^{3} 5x^2 \Big[y \Big]_{x^2}^{9} dx = \int_{-3}^{3} (45x^2 - 5x^4)\, dx$$
$$= [15x^3 - x^5]_{-3}^{3} = 2(15 \cdot 3^3 - 3^5) = 324$$

9. Compute $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S}$ over the complete surface of the cylinder $x^2 + y^2 \le 9$ for $0 \le z \le 4$ oriented out from the cylinder for $\vec{F} = (xz, yz, z^2)$.
Hint: Use Gauss' Theorem.

a. 24π
b. 36π
c. 72π
d. 144π
e. 288π Correct Choice

Solution: By Gauss' Theorem, $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S} = \iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\, dV$ and $\vec{\nabla} \cdot \vec{F} = z + z + 2z = 4z$

$$\iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\, dV = \int_0^4 \int_0^{2\pi} \int_0^3 4zr\, dr\, d\theta\, dz = 4\left[\frac{z^2}{2} \right]_0^4 (2\pi) \left[\frac{r^2}{2} \right]_0^3 = 4(8)(2\pi)\left(\frac{9}{2} \right) = 288\pi$$

10. Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx$ in problem (11).
Select its value here:

a. $\frac{1}{4}\sin 2$

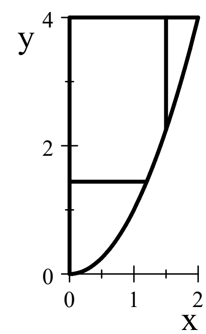
b. $\frac{1}{4}\sin 4$

c. $\frac{1}{2}\sin 4$

d. $\frac{1}{4}\sin 16$ Correct Choice

e. $\frac{1}{2}\sin 16$

Solution: Reverse the order of integration:



$$\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} x\cos(y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^2}{2}\cos(y^2) \right]_{x=0}^{\sqrt{y}} dy = \int_0^4 \frac{y}{2}\cos(y^2)\,dy$$

Substitute: $u = y^2$ $\quad du = 2y\,dy$ $\quad y\,dy = \frac{1}{2}du$

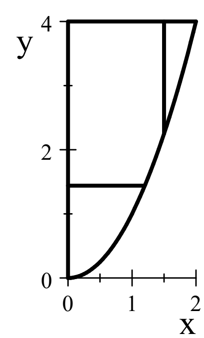
$$\int_0^2 \int_{x^2}^4 x\cos(y^2)\,dy\,dx = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u)$$

$$= \frac{1}{4}\sin(y^2)\Big|_0^4 = \frac{1}{4}\sin 16$$

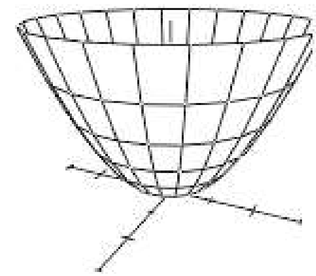
Work Out: (Points indicated. Part credit possible. Show all work.)

11. (5 points) Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^4 x\cos(y^2)$.
Shade in the region.
Compute its value in problem (10).

Solution:



12. (20 points) Verify Stokes' Theorem $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (-yz, xz, z^2)$ and the **surface** which is the piece of the paraboloid P given by $z = x^2 + y^2$ between $z = 1$ and $z = 9$ oriented up and in.

Notice that the boundary of P is two circles.

Be sure to check orientations. Use the following steps:

a. The paraboloid may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$ for $1 \le r \le 3$.

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\cos\theta, & \sin\theta, & 2r) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0\ ) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ has the correct orientation.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(y) + \hat{k}(z+z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (-r\cos\theta, -r\sin\theta, 2r^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 2r^3\cos^2\theta + 2r^3\sin^2\theta + 2r^3 = 4r^3$$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_P \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_1^3 4r^3\, dr\, d\theta = 2\pi[r^4]_{r=1}^3 = 2\pi(81-1) = 160\pi$$

Recall $\vec{F} = (-yz, xz, z^2)$

b. Parametrize the upper circle U and compute the line integral.

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 9)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$ oriented correctly counterclockwise

$\vec{F}\big|_{\vec{r}(\theta)} = (-27\sin\theta, 27\cos\theta, 81)$

$$\oint_U \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 81\sin^2\theta + 81\cos^2\theta\, d\theta = \int_0^{2\pi} 81\, d\theta = 162\pi$$

c. Parametrize the lower circle L and compute the line integral.

$\vec{r}(\theta) = (\cos\theta, \sin\theta, 1)$

$\vec{v}(\theta) = (-\sin\theta, \cos\theta, 0)$ oriented counterclockwise, need clockwise

Rev $\vec{v}(\theta) = (\sin\theta, -\cos\theta, 0)$

$\vec{F}\big|_{\vec{r}(\theta)} = (-\sin\theta, \cos\theta, 1)$

$$\oint_L \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -\sin^2\theta - \cos^2\theta\, d\theta = -\int_0^{2\pi} 1\, d\theta = -2\pi$$

d. Combine $\oint_U \vec{F} \cdot d\vec{s}$ and $\oint_L \vec{F} \cdot d\vec{s}$ to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$

$$\oint_{\partial P} \vec{F} \cdot d\vec{s} = \oint_U \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 162\pi - 2\pi = 160\pi$$

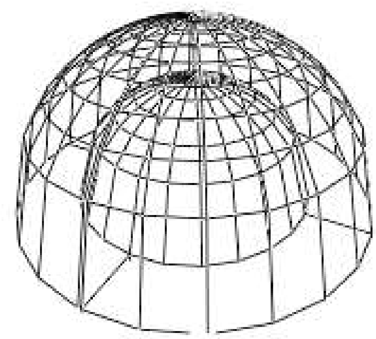
which agrees with part (a).

13. (15 points) (Also replaces Exam 3 #12.)

Find the mass of the solid between the hemispheres

$z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{9 - x^2 - y^2}$

for $z \geq 0$ if the density is $\delta = \dfrac{1}{x^2 + y^2 + z^2}$.



Solution: In spherical coordinates, $2 \leq \rho \leq 3$ and $\delta = \dfrac{1}{\rho^2}$ and $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

$$M = \iiint \delta \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_2^3 \frac{1}{\rho^2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \Big[\rho \Big]_2^3 \Big[-\cos\varphi \Big]_0^{\pi/2}$$

$$= 2\pi(3 - 2)(-0 - -1) = 2\pi$$

14. (15 points) (Also replaces Exam 3 #13.)

Find the **centroid** of the **solid** inside

the paraboloid $z = x^2 + y^2$ for $1 \leq z \leq 2$.

Hint: Put the differentials in the order $dr \, dz \, d\theta$.

Solution: In cylindrical coordinates, $dV = r \, dr \, d\theta \, dz$ and the paraboloid is $z = r^2$ or $r = \sqrt{z}$.

The volume is:

$$V = \iiint 1 \, dV = \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{z}} r \, dr \, dz \, d\theta = 2\pi \int_1^2 \Big[\frac{r^2}{2} \Big]_0^{\sqrt{z}} dz = \pi \int_1^2 z \, dz = \pi \Big[\frac{z^2}{2} \Big]_1^2$$

$$= \frac{\pi}{2}(4 - 1) = \frac{3\pi}{2}$$

The z-moment is:

$$V_z = \iiint z \, dV = \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{z}} z \, r \, dr \, dz \, d\theta = 2\pi \int_1^2 z \Big[\frac{r^2}{2} \Big]_0^{\sqrt{z}} dz$$

$$= \pi \int_1^2 z^2 \, dz = \pi \Big[\frac{z^3}{3} \Big]_1^2 = \frac{\pi}{3}(8 - 1) = \frac{7\pi}{3}$$

So the z-component of the centroid is:

$$\bar{z} = \frac{V_z}{V} = \frac{7\pi}{3} \frac{2}{3\pi} = \frac{14}{9} \approx 1.444$$

We know $\bar{x} = \bar{y} = 0$ by symmetry. So the centriod is $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right)$.