Definition and Properties of Determinants

Definitions:

If A is an $n \times n$ matrix, then the determinant of A, denoted by either $\det A$ or $|A|$, is defined by

$$\det A = |A| = \sum_{\text{perm } p} \varepsilon_p A_{1p_1} \cdots A_{ip_i} \cdots A_{np_n}$$

where ε_p is the sign of the permutation p given by

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \text{ is even} \\ -1 & \text{if } p \text{ is odd} \end{cases}$$

Row Notation: If $\vec{v}_1, \cdots, \vec{v}_n$ are n vectors in $\mathbb{R}^n$ then $\det(\vec{v}_1, \cdots, \vec{v}_n)$ is the determinant of the matrix whose rows are $\vec{v}_1, \cdots, \vec{v}_n$; i.e.

$$\det(\vec{v}_1, \cdots, \vec{v}_n) = \det\begin{pmatrix} \leftarrow & \vec{v}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{v}_n & \rightarrow \end{pmatrix}$$

Properties:

1. **Transpose**:
$$\det A^{\mathsf{T}} = \det A$$
   - Every theorem below involving rows can be restated in terms of columns.

2. **Triangular**:
If A is triangular (or diagonal), then $\det A$ is the product of the diagonal entries.

3. **Row of zeros**:
$\det\left(\vec{v}_1, \cdots, \vec{0}, \cdots, \vec{v}_n\right) = 0$

4. **Interchange rows**:
$\det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{w}, \cdots, \vec{v}_n) = -\det(\vec{v}_1, \cdots, \vec{w}, \cdots, \vec{u}, \cdots, \vec{v}_n)$ .... (Row Operation I)

5. **Two equal rows**:
$\det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{u}, \cdots, \vec{v}_n) = 0$

6. **Multiple of row**:
$\det(\vec{v}_1, \cdots, c\vec{u}, \cdots, \vec{v}_n) = c \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{v}_n)$ .... (Row Operation II)

7. **Addition in row**:
$\det(\vec{v}_1, \cdots, \vec{u} + \vec{w}, \cdots, \vec{v}_n) = \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{v}_n) + \det(\vec{v}_1, \cdots, \vec{w}, \cdots, \vec{v}_n)$

8. **Add multiple of one row to another row**:
$\det(\vec{v}_1, \cdots, \vec{u} + c\vec{w}, \cdots, \vec{w}, \cdots, \vec{v}_n) = \det(\vec{v}_1, \cdots, \vec{u}, \cdots, \vec{w}, \cdots, \vec{v}_n)$ .... (Row Operation III)

9. **Multiple of matrix**:
$\det(cA) = c^n \det A$

10. **Product of Matrices**:
$\det(AB) = \det A \det B$

11. **Determinant of inverse**:
If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

12. **Invertibility**:
$\det A \neq 0 \iff A$ is invertible (non-singular) $\iff A\vec{x} = B$ has a unique solution $\iff N(A) = \{\vec{0}\}$
$\det A = 0 \iff A$ is non-invertible (singular) $\iff A\vec{x} = B$ has no solution or ∞-many solutions $\iff N(A) \neq \{\vec{0}\}$