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1-8	/24
9	/ 6
10	/10
11	/10

MATH 253 EXAM 1 Fall 1998

Sections 501-503 P. Yasskin

Multiple Choice: (3 points each)

Problems 1 – 3: Find the plane tangent to the graph of the equation $z\ln(y) - z\ln(x) = e^2$ at the point $(x,y,z) = (1,e,e^2)$. Write the equation of the plane in the form $z = Ax + By + C$ and find the values of A, B and C in problems 1, 2 and 3:

Solution: $F(x,y,z) = z\ln(y) - z\ln(x)$

METHOD 1:

$P = (1,e,e^2)$ $\quad \vec{\nabla}F = \left(-\frac{z}{x}, \frac{z}{y}, \ln(y) - \ln(x)\right)$ $\quad \vec{N} = \vec{\nabla}F(1,e,e^2) = (-e^2, e, 1)$ $\quad X = (x,y,z)$

Plane: $\vec{N} \bullet X = \vec{N} \bullet P$ $\quad -e^2x + ey + z = -e^2 1 + ee + e^2 = e^2$ $\quad z = e^2x - ey + e^2$

METHOD 2:

$F_x(x,y,z) = -\frac{z}{x}$ $\quad F_y(x,y,z) = \frac{z}{y}$ $\quad F_z(x,y,z) = \ln(y) - \ln(x)$

$F_x(1,e,e^2) = -e^2$ $\quad F_y(1,e,e^2) = e$ $\quad F_z(1,e,e^2) = 1$

$F_x(1,e,e^2)(x-1) + F_y(1,e,e^2)(y-e) + F_z(1,e,e^2)(z-e^2) = 0$

$-e^2(x-1) + e(y-e) + 1(z-e^2) = 0$ $\quad -e^2x + e^2 + ey - e^2 + z - e^2 = 0$ $\quad z = e^2x - ey + e^2$

In either case: $A = e^2$ $\quad B = -e$ $\quad C = e^2$

1. $A =$

- **a**. 1
- **b**. e
- **c**. $e^2 \ldots$ correctchoice
- **d**. $-e$
- **e**. $-e^2$

2. $B =$

- **a**. 1
- **b**. e
- **c**. e^2
- **d**. $-e \ldots$ correctchoice
- **e**. $-e^2$

3. $C =$

- **a**. 1
- **b**. e
- **c**. $e^2 \ldots$ correctchoice
- **d**. $-e$
- **e**. $-e^2$

Problems 4 – 6: Find the plane tangent to the graph of the function $f(x,y) = \dfrac{x-y}{x^2+y^2}$ at the point $(x,y) = (2,1)$. Write the equation of the plane in the form $z = Ax + By + C$ and find the values of A, B and C in problems 4, 5 and 6:

Solution: $f(x,y) = \dfrac{x-y}{x^2+y^2}$ $\quad f(2,1) = \dfrac{2-1}{4+1} = \dfrac{1}{5}$

$f_x(x,y) = \dfrac{(x^2+y^2)1-(x-y)2x}{(x^2+y^2)^2}$ $\quad f_x(2,1) = \dfrac{(4+1)1-(2-1)4}{(4+1)^2} = \dfrac{1}{25}$

$f_y(x,y) = \dfrac{(x^2+y^2)(-1)-(x-y)2y}{(x^2+y^2)^2}$ $\quad f_y(2,1) = \dfrac{(4+1)(-1)-(2-1)2}{(4+1)^2} = \dfrac{-7}{25}$

$$z = f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1) = \frac{1}{5}+\frac{1}{25}(x-2)-\frac{7}{25}(y-1)$$

$$= \frac{1}{25}x-\frac{7}{25}y+\frac{1}{5}+\frac{-2}{25}+\frac{7}{25} = \frac{1}{25}x-\frac{7}{25}y+\frac{2}{5}$$

So $A = \dfrac{1}{25}$ $\quad B = -\dfrac{7}{25}$ $\quad C = \dfrac{2}{5}$

4. $A =$

- **a.** $\frac{1}{25}\ldots$ correctchoice
- **b.** $\frac{7}{25}$
- **c.** $-\frac{1}{5}$
- **d.** $-\frac{1}{25}$
- **e.** $-\frac{7}{25}$

5. $B =$

- **a.** $\frac{1}{25}$
- **b.** $\frac{7}{25}$
- **c.** $-\frac{1}{5}$
- **d.** $-\frac{1}{25}$
- **e.** $-\frac{7}{25}\ldots$ correctchoice

6. $C =$

- **a.** $\frac{2}{5}\ldots$ correctchoice
- **b.** $\frac{1}{5}$
- **c.** 0
- **d.** $-\frac{1}{5}$
- **e.** $-\frac{2}{5}$

7. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm and 50 cm with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in the calculated surface area of the box.

Solution: $A = 2xy + 2xz + 2yz$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz = (2y + 2z)dx + (2x + 2z)dy + (2x + 2y)dz$$

$$= (120 + 100).2 + (160 + 100).2 + (160 + 120).2 = 152$$

- **a**. 38 cm^2
- **b**. 95 cm^2
- **c**. 144 cm^2
- **d**. 148 cm^2
- **e**. 152 cm^2… correctchoice

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8. Consider the function $f(x,y) = 3(x + y)^3$. Find the set of **all** points (x,y) where $\vec{\nabla} f = 0$.

Solution: $\vec{\nabla} f = \left(9(x+y)^2, 9(x+y)^2\right) = 0$ when $x + y = 0$ which is the line $y = -x$ which may be parametrized as $(x,y) = (t,-t)$.

- **a**. The point $(x,y) = (0,0)$.
- **b**. The point $(x,y) = (1,-1)$.
- **c**. The line $(x,y) = (t,-t)$.… correctchoice
- **d**. The line $(x,y) = (t,t)$.
- **e**. The circle $x^2 + y^2 = \frac{1}{9}$.

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9. (6 points) Suppose $p = p(x,y)$, while $x = x(u,v)$ and $y = y(u,v)$. Further, you know the following information:

$x(1,2) = 3$	$y(1,2) = 4$	$p(1,2) = 5$	$p(3,4) = 6$
$\frac{\partial p}{\partial x}(1,2) = 7$	$\frac{\partial p}{\partial y}(1,2) = 8$	$\frac{\partial p}{\partial x}(3,4) = 9$	$\frac{\partial p}{\partial y}(3,4) = 10$
$\frac{\partial x}{\partial u}(1,2) = 11$	$\frac{\partial x}{\partial v}(1,2) = 12$	$\frac{\partial x}{\partial u}(3,4) = 13$	$\frac{\partial x}{\partial v}(3,4) = 14$
$\frac{\partial y}{\partial u}(1,2) = 15$	$\frac{\partial y}{\partial v}(1,2) = 16$	$\frac{\partial y}{\partial u}(3,4) = 17$	$\frac{\partial y}{\partial v}(3,4) = 18$

Write out the chain rule for $\frac{\partial p}{\partial v}$. Then use it and the above information to compute $\frac{\partial p}{\partial v}(1,2)$.

Solution: $\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial v} = \left.\frac{\partial p}{\partial x}\right|_{(x(u,v),y(u,v))}\frac{\partial x}{\partial v} + \left.\frac{\partial p}{\partial y}\right|_{(x(u,v),y(u,v))}\frac{\partial y}{\partial v}$

$$\frac{\partial p}{\partial v}(1,2) = \left.\frac{\partial p}{\partial x}\right|_{(x(1,2),y((1,2)))}\frac{\partial x}{\partial v}(1,2) + \left.\frac{\partial p}{\partial y}\right|_{(x((1,2)v),y((1,2)))}\frac{\partial y}{\partial v}(1,2)$$

$$= \frac{\partial p}{\partial x}(3,4)\frac{\partial x}{\partial v}(1,2) + \frac{\partial p}{\partial y}(3,4)\frac{\partial y}{\partial v}(1,2) = 9 \bullet 12 + 10 \bullet 16 = 268$$

10. (10 points) Find all critical points of the function $f(x,y) = xy^2 + 3x^2 - 3y^2 - 2x^3$ and classify each as a local maximum, a local minimum or a saddle point.

Solution: $f_x = y^2 + 6x - 6x^2 = 0 \qquad f_y = 2xy - 6y = 2y(x-3) = 0$

From f_y: $\quad y = 0 \quad$ or $\quad x = 3$

Case: $y = 0$: From f_x: $\quad 6x - 6x^2 = 6x(1-x) = 0 \quad$ So: $x = 0 \quad$ or $\quad x = 1$.

Case: $x = 3$: From f_x: $\quad y^2 + 18 - 54 = 0 \quad y^2 = 36 \quad$ So: $y = 6 \quad$ or $\quad y = -6$.

So the critical points are: $\quad (0,0) \quad (1,0) \quad (3,6) \quad (3,-6)$

To classify: $\quad f_{xx} = 6 - 12x \qquad f_{yy} = 2x - 6 \qquad f_{xy} = 2y \qquad D = f_{xx}f_{yy} - f_{xy}{}^2$

x	y	f_{xx}	f_{yy}	f_{xy}	D	Classification
0	0	6	-6	0	$-36 < 0$	saddle
1	0	$-6 < 0$	-4	0	$24 > 0$	local maximum
3	6	-30	0	12	$-144 < 0$	saddle
3	-6	-30	0	-12	$-144 < 0$	saddle

11. (10 points) Find the dimensions and volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $3x + 2y + z = 6$.

Solution: Maximize $\quad V = xyz \quad$ subject to $\quad g = 3x + 2y + z = 6$.

METHOD 1: Eliminate a variable: $\quad z = 6 - 3x - 2y$

$V = xy(6 - 3x - 2y) = 6xy - 3x^2y - 2xy^2$

$V_x = 6y - 6xy - 2y^2 = y(6 - 6x - 2y) = 0 \qquad V_y = 6x - 3x^2 - 4xy = x(6 - 3x - 4y) = 0$

To have a non-zero volume, we must have $x \neq 0$ and $y \neq 0$.

So $\left\{ \begin{matrix} 6 - 6x - 2y = 0 \\ 6 - 3x - 4y = 0 \end{matrix} \right\}$ or $\left\{ \begin{matrix} 6x + 2y = 6 \\ 3x + 4y = 6 \end{matrix} \right\}$ or $\left\{ \begin{matrix} 12x + 4y = 12 \\ 3x + 4y = 6 \end{matrix} \right\}$

Subtracting, we have: $9x = 6$ or $x = \frac{2}{3}$. So $3\left(\frac{2}{3}\right) + 4y = 6$ or $4y = 4$ or $y = 1$.

So $z = 6 - 3\left(\frac{2}{3}\right) - 2(1) = 2$

So the dimensions are $\quad x = \frac{2}{3}, \quad y = 1 \quad$ and $\quad z = 2$. The volume is $\quad V = \frac{4}{3}$.

METHOD 2: Lagrange Multipliers: $\quad \vec{\nabla} V = (yz, xz, xy) \qquad \vec{\nabla} g = (3, 2, 1)$

$\vec{\nabla} V = \lambda \vec{\nabla} g \qquad (yz, xz, xy) = \lambda(3, 2, 1)$

Solve $\quad yz = 3\lambda \quad xz = 2\lambda \quad xy = \lambda \quad$ and $\quad 3x + 2y + z = 6 \quad$ for $\quad x, y, z$ and λ.

$\lambda = xy = \frac{yz}{3} = \frac{xz}{2} \qquad x = \frac{z}{3} \qquad y = \frac{z}{2} \qquad 3\left(\frac{z}{3}\right) + 2\left(\frac{z}{2}\right) + z = 6 \qquad 3z = 6$

$z = 2 \qquad x = \frac{2}{3} \qquad y = 1 \qquad$ The volume is $\quad V = \frac{4}{3}$.