Name__________________________ ID______________ Section______

MATH 253 FINAL EXAM Fall 1998

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (10 points each)

HINTS: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

1. Compute $\lim_{n\to\infty} \frac{3n^2}{1+n^3}$

a. 0 correctchoice
b. 1
c. 2
d. 3
e. Divergent

$$\lim_{n\to\infty} \frac{3n^2}{1+n^3} = \lim_{n\to\infty} \frac{3n^2}{1+n^3}\frac{\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{1}{n^3}+1} = 0$$

2. Find r such that $5 + 5r + 5r^2 + 5r^3 + 5r^4 + \cdots = 3.$

a. $\frac{2}{5}$
b. $-\frac{2}{5}$
c. $\frac{3}{5}$
d. $\frac{5}{3}$
e. $-\frac{2}{3}$ correctchoice

$$\frac{5}{1-r} = 3 \qquad \frac{5}{3} = 1 - r \qquad r = 1 - \frac{5}{3} = -\frac{2}{3}$$

3. Compute $\sum_{k=1}^{99}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$

a. $.9$ correctchoice

b. $.99$

c. 1

d. 1.1

e. Divergent

$$\sum_{k=1}^{99}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)=\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)+\cdots+\left(\frac{1}{\sqrt{99}}-\frac{1}{\sqrt{100}}\right)$$
$$=1-\frac{1}{10}=.9$$

4. The series $\sum_{n=1}^{\infty}\frac{1}{n^2+\sqrt{n}}$ is

a. divergent by comparison to $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

b. convergent by comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$. correctchoice

c. divergent by the ratio test.

d. convergent by the ratio test.

e. divergent by the n^{th}-term test.

In the series $\sum_{n=1}^{\infty}\frac{1}{n^2+\sqrt{n}}$ the largest term in the denominator is n^2. So, we apply the Comparison Test by comparing with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is a convergent p-series since $p=2>1$. Since $n^2+\sqrt{n}>n^2$ we have $\frac{1}{n^2+\sqrt{n}}<\frac{1}{n^2}$. So $\sum_{n=1}^{\infty}\frac{1}{n^2+\sqrt{n}}$ is also convergent.

5. The series $\sum_{n=1}^{\infty}(-1)^n \frac{3n^2}{1+n^3}$ is

a. absolutely convergent.

b. conditionally convergent. correctchoice

c. divergent to ∞.

d. divergent to $-\infty$.

e. oscillatory divergent.

The series $\sum_{n=1}^{\infty}(-1)^n \frac{3n^2}{1+n^3}$ is convergent because it is an alternating, decreasing series and $\lim_{n\to\infty} \frac{3n^2}{1+n^3} = 0$. The related absolute series is $\sum_{n=1}^{\infty} \frac{3n^2}{1+n^3}$ which is divergent by the Integral Test since $\int_1^{\infty} \frac{3x^2}{1+x^3}\,dx = \left[\ln(1+x^3)\right]_1^{\infty} = \infty$. So $\sum_{n=1}^{\infty}(-1)^n \frac{3n^2}{1+n^3}$ is conditionally convergent.

6. Compute $\lim_{x\to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4}$

a. 0

b. $\frac{1}{24}$

c. $\frac{1}{12}$

d. $\frac{2}{3}$ correctchoice

e. ∞

$$\lim_{x\to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} = \lim_{x\to 0} \frac{\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right] - 1 + 2x^2}{x^4}$$

$$= \lim_{x\to 0} \frac{\frac{(2x)^4}{4!} - \cdots}{x^4} = \frac{16}{24} = \frac{2}{3}$$

7. Given that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (for $|x| < 1$), then (for $|x| < 1$) we have $\sum_{n=0}^{\infty} n x^n =$

a. $\frac{1}{1-x}$

b. $\frac{1}{(1-x)^2}$

c. $\frac{x}{(1-x)^2}$ correctchoice

d. $\frac{x}{1-x}$

e. $\frac{n}{1-x}$

Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Apply $\frac{d}{dx}$ to get $\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$.

Multiply by x to get $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$.

8. Find the x-coordinate of the center of mass of the rectangle $0 \le x \le 3$, $0 \le y \le 2$ if the density is $\rho = xy$.

a. .5

b. 1

c. 1.5

d. 2 correctchoice

e. 2.5

$$M = \iint \rho \, dA = \int_0^2 \int_0^3 xy \, dx \, dy = \left[\frac{x^2}{2} \right]_0^3 \left[\frac{y^2}{2} \right]_0^2 = \frac{9}{2} \bullet \frac{4}{2} = 9$$

$$x\text{-mom} = \iint x\rho \, dA = \int_0^2 \int_0^3 x^2 y \, dx \, dy = \left[\frac{x^3}{3} \right]_0^3 \left[\frac{y^2}{2} \right]_0^2 = \frac{27}{3} \bullet \frac{4}{2} = 18$$

$$\bar{x} = \frac{x\text{-mom}}{M} = \frac{18}{9} = 2$$

9. Find the mass of the solid inside the cylinder $x^2 + y^2 = 1$ above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2$ if the density is $\rho = x^2 + y^2$.

a. $\frac{\pi}{2}$

b. $\frac{2\pi}{3}$ correctchoice

c. π

d. $\frac{3\pi}{2}$

e. 2π

In cylindrical coordinates, the cylinder is $r = 1$ the paraboloid is $z = r^2$ and the density is $\rho = r^2$.

$$M = \iiint \rho \, dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^2 r^2 \; r \; dz \, dr \, d\theta = 2\pi \int_0^1 \Big[r^3 z \Big]_{r^2}^2 dr = 2\pi \int_0^1 r^3 (2 - r^2) \, dr$$

$$= 2\pi \left[\frac{2r^4}{4} - \frac{r^6}{6} \right]_0^1 = 2\pi \left[\frac{1}{2} - \frac{1}{6} \right] = \frac{2\pi}{3}$$

10. If $F = (xz^2, -yz^2, z^3)$ then $\vec{\nabla} \bullet \vec{F} =$

a. $(z^2, z^2, 3z^2)$

b. $(z^2, -z^2, 3z^2)$

c. $3z^2$ correctchoice

d. $5z^2$

e. $2(y - x)z$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(z^3) = z^2 - z^2 + 3z^2 = 3z^2$$

11. If $F = (xz^2, -yz^2, z^3)$ then $\vec{\nabla} \times \vec{F} =$

a. $2(y - x)z$

b. $2(x + y)z$

c. $(2yz, -2xz, 0)$

d. $(2yz, 2xz, 0)$ correctchoice

e. $\vec{0}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz^2 & -yz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - -2yz) - \hat{\jmath}(0 - 2xz) + k(0 - 0) = (2yz, 2xz, 0)$$

12. Compute $\int_{\vec{r}(t)} y^2 e^{xy}\,dx + (1+xy)e^{xy}\,dy$ along the spiral $\vec{r}(t) = (t\cos t, t\sin t)$ from $t = \pi$ to $t = 3\pi$.

HINT: Find a scalar potential for $\vec{F} = (y^2 e^{xy}, (1+xy)e^{xy})$.

a. -2π

b. 0 correctchoice

c. π

d. 3π

e. 4π

$\frac{\partial f}{\partial x} = y^2 e^{xy} \quad \Rightarrow \quad f = ye^{xy} + g(y) \quad \Rightarrow \quad \frac{\partial f}{\partial y} = e^{xy} + xye^{xy} + \frac{dg}{dy} = (1+xy)e^{xy} + \frac{dg}{dy}$

$\frac{\partial f}{\partial y} = (1+xy)e^{xy} \quad \Rightarrow \quad \frac{dg}{dy} = 0 \quad \Rightarrow \quad g = C$ Take $C = 0$.

So $f = ye^{xy}$

$\int_{\vec{r}(t)} y^2 e^{xy}\,dx + (1+xy)e^{xy}\,dy = f(\vec{r}(3\pi)) - f(\vec{r}(\pi)) = f(-3\pi, 0) - f(-\pi, 0) = 0e^0 - 0e^0 = 0$

Work-Out Problems

13. (25 points) You are given: $e^{(-x^2)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$.

a. (10 pt) If $f(x) = e^{(-x^2)}$, find $f^{(6)}(0)$.

The term with an x^6 is $\frac{f^{(6)}(0)}{6!}x^6 = -\frac{x^6}{6}$. So:
$f^{(6)}(0) = -\frac{6!}{6} = -5! = -120.$

b. (10 pt) Use the **quadratic** Taylor polynomial approximation about $x = 0$ for $e^{(-x^2)}$ to estimate $\int_0^{0.1} e^{(-x^2)}\,dx$. (Keep 8 digits.)

The quadratic Taylor polynomial approximation is $e^{(-x^2)} \approx 1 - x^2$. So:

$$\int_0^{0.1} e^{(-x^2)}\,dx \approx \int_0^{0.1} 1 - x^2\,dx = \left[x - \frac{x^3}{3}\right]_0^{0.1} = .1 - \frac{.001}{3} = .09966667$$

c. (5 pt) Your result in (b) is equal to $\int_0^{0.1} e^{(-x^2)}\,dx$ to within ± how much? Why?

Since $e^{(-x^2)}$ and $\int_0^{0.1} e^{(-x^2)}\,dx$ are alternating decreasing series, the error is at most the next term:

$$\int_0^{0.1} \frac{x^4}{2}\,dx = \left[\frac{x^5}{10}\right]_0^{0.1} = \frac{(.1)^5}{10} = 10^{-6} = .000001$$

14. (15 points) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{3^n n^3}$.

Be sure to identify each of the following and give reasons:

(1 pt) Center of Convergence: $a = \underline{\quad 5 \quad}$

$a_n = \frac{(x-5)^n}{3^n n^3}$ $\qquad$ $a_{n+1} = \frac{(x-5)^{n+1}}{3^{n+1}(n+1)^3}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{3^{n+1}(n+1)^3}\frac{3^n n^3}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)}{3}\left(\frac{n}{n+1}\right)^3\right| = \frac{|x-5|}{3}$$

The series converges if $\frac{|x-5|}{3} < 1$ or $|x-5| < 3$

Radius of Convergence: $R = \underline{\quad 3 \quad}$ (5 pt)

(1 pt) Right Endpoint: $x = \underline{\quad 5+3 = 8 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(8-5)^n}{3^n n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is s p-series with $p = 3 > 1$.

At the Right Endpoint the Series { **[Converges]** / Diverges } (circle one) (3 pt)

(1 pt) Left Endpoint: $x = \underline{\quad 5-3 = 2 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(2-5)^n}{3^n n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges because its related absolute series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges OR because it is an alternating decreasing series and $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

At the Left Endpoint the Series { **[Converges]** / Diverges } (circle one) (3 pt)

(1 pt) Interval of Convergence: $\underline{\quad 2 \le x \le 8 \quad \text{or} \quad [2,8] \quad}$

15. (30 points) Stokes' Theorem states that if S is a surface in 3-space and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S then

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s}$$

Verify Stokes' Theorem if $F = (-yz, xz, z^2)$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ below $z = 2$ with normal pointing in and up.

a. (5 pt) Compute $\vec{\nabla} \times \vec{F}$. (HINT: Use rectangular coordinates.)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

b. (10 pt) Compute $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$.

(HINT: Here is the parametrization of the cone and the steps you should use. Remember to check the orientation of the surface.)

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$\vec{R}_r = (\cos\theta, \sin\theta, 1)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = i(-r\cos\theta) - j(r\sin\theta) + k(r) = (-r\cos\theta, -r\sin\theta, r)$

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (-r\cos\theta, -r\sin\theta, 2r)$

$\left(\vec{\nabla} \times \vec{F}\right) \bullet \vec{N} = r^2\cos^2\theta + r^2\sin^2\theta + 2r^2 = 3r^2$

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \int_0^{2\pi} \int_0^2 3r^2\, dr\, d\theta = 2\pi[r^3]_0^2 = 16\pi$$

c. (15 pt) Compute $\oint_{\partial S} \vec{F} \bullet d\vec{s}$. Recall $F = (-yz, xz, z^2)$.

(HINT: Parametrize the boundary circle. Remember to check the orientation of the curve.)

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$ $\qquad$ $\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (-4\sin\theta, 4\cos\theta, 4)$ $\qquad$ $\vec{F} \bullet \vec{v} = 8\sin^2\theta + 8\cos^2\theta = 8$

$$\oint_{\partial S} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} 8\, d\theta = 16\pi$$

16. (20 points) Find the minimum value and its location(s) for the function $f(x,y) = xy$ on the ellipse $9x^2 + 4y^2 = 72$.

$f = xy \qquad g = 9x^2 + 4y^2 - 72 \qquad \vec{\nabla} f = (y,x) \qquad \vec{\nabla} g = (18x, 8y)$

$$\vec{\nabla} f = \lambda \vec{\nabla} g: \quad \left\{ \begin{array}{l} y = \lambda 18x \;\Rightarrow\; \lambda = \dfrac{y}{18x} \\ x = \lambda 8y \;\Rightarrow\; \lambda = \dfrac{x}{8y} \end{array} \right\} \quad \Rightarrow \quad \frac{y}{18x} = \frac{x}{8y} \quad \Rightarrow \quad 4y^2 = 9x^2$$

Constraint: $9x^2 + 4y^2 = 72 \qquad 18x^2 = 72 \qquad x^2 = 4 \qquad x = \pm 2$

$y = \pm \frac{3}{2} x = \pm \frac{3}{2}(\pm 2) = \pm 3$

Critical points: $(2,3), \quad (2,-3), \quad (-2,3), \quad (-2,-3)$

$f(2,3) = 6, \quad f(2,-3) = -6, \quad f(-2,3) = -6, \quad f(-2,-3) = 6$

The minimum is -6 at $(x,y) = (2,-3)$ and $(x,y) = (-2,3)$.