1-11	/55
12	/10
13	/30
14	/12
Total	/107

Name________________________

MATH 253 Final Exam Fall 2016

Sections 201/202 Solutions P. Yasskin

Multiple Choice: (5 points each.)

Two questions have part credit, but I won't say which.

1. Find the tangential acceleration along the curve $\vec{r}(t) = (3t^2, 4t^3, 3t^4)$.

a. $6 - 36t^2$

b. $6 + 36t^2$ Correct Choice

c. $6 - 24t^2$

d. $6 + 24t^2$

e. $12\sqrt{3 + 4t^2 + 9t^4}$

$\vec{v} = (6t, 12t^2, 12t^3) \qquad |\vec{v}| = \sqrt{36t^2 + 144t^4 + 144t^6} = 6t + 12t^3$

$a_N = \dfrac{d|\vec{v}|}{dt} = \dfrac{d}{dt}(6t + 12t^3) = 6 + 36t^2$

2. The graphs of $z = x^2 + y^3$ and $z = 129x^2 - y^3$ intersect in the curve $\vec{r}(t) = (x(t), y(t), z(t))$. If $x(t) = t^3$, then $z(t) =$

a. $63t^6$

b. $64t^6$

c. $65t^6$ Correct Choice

d. $127t^6$

e. $128t^6$

$x^2 + y^3 = 129x^2 - y^3 \quad \Rightarrow \quad 2y^3 = 128x^2 \quad \Rightarrow \quad y^3 = 64x^2 \quad \Rightarrow \quad y = 4x^{2/3}$

If $x = t^3$, then $y = 4t^2$ and $z = x^2 + y^3 = t^6 + 64t^6 = 65t^6$.

We check with $z = 129x^2 - y^3 = 129t^6 - 64t^6 = 65t^6$.

3. Antwoman is currently running across a frying pan. She is currently at the point $(3, 2)$ and has speed 20 cm/sec in the direction $\left(\frac{3}{5}, \frac{4}{5}\right)$. She measures the temperature to be 325°K and its gradient to be $(-5, 2)$ °K/cm. At what rate (in °K/sec) does she see the temperature changing?

a. -28 Correct Choice

b. -14

c. -1.4

d. 1.4

e. 28

The gradient of the temperature is $\vec{\nabla}T = (-5, 2)$ and her velocity is $\vec{v} = 20\left(\frac{3}{5}, \frac{4}{5}\right) = (12, 16)$.

So the rate of change along the curve is $\dfrac{dT}{dt} = \vec{v} \cdot \vec{\nabla}T = (12, 16) \cdot (-5, 2) = -28$.

4. Find 3 positive numbers x, y and z, whose sum is 90 such that $f(x,y,z) = xy^2z^3$ is a maximum. What is xyz?

a. $2^3 \cdot 3 \cdot 5^3 \cdot 7$

b. $2^2 \cdot 3^2 \cdot 5^4$

c. $2 \cdot 3^4 \cdot 5^3$ Correct Choice

d. $2^2 \cdot 5^3 \cdot 7^2$

e. $2 \cdot 3 \cdot 5^4 \cdot 7$

METHOD 1: Lagrange Multipliers: $x+y+z=90$

$f = xy^2z^3 \qquad \vec{\nabla}f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x+y+z \qquad \vec{\nabla}g = (1,1,1)$

$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad y^2z^3 = \lambda, \quad 2xyz^3 = \lambda, \quad 3xy^2z^2 = \lambda \quad \Rightarrow \quad y^2z^3 = 2xyz^3, \quad y^2z^3 = 3xy^2z^2$

$\Rightarrow \quad y = 2x, \quad z = 3x \quad \Rightarrow \quad x+y+z = x+2x+3x = 90 \quad \Rightarrow \quad 6x = 90 \quad \Rightarrow \quad x = 15$

$x = 15, \quad y = 30, \quad z = 45 \qquad xyz = 15 \cdot 30 \cdot 45 = 2 \cdot 3^4 \cdot 5^3$

METHOD 2: Eliminate a Variable: $x+y+z=90$

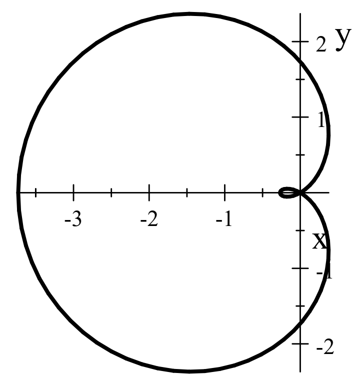
$x = 90-y-z \quad \Rightarrow \quad f = (90-y-z)y^2z^3 = 90y^2z^3 - y^3z^3 - y^2z^4$

$$\begin{matrix} f_y = 180yz^3 - 3y^2z^3 - 2yz^4 = 0 \\ f_z = 270y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 180 - 3y - 2z = 0 \\ 270 - 3y - 4z = 0 \end{matrix}$$

Subtract: $90 - 2z = 0 \quad \Rightarrow \quad z = 45$ Substitute back: $180 - 3y - 90 = 0 \quad \Rightarrow \quad y = 30$

Substitute back: $x = 90-y-z = 90-30-45 = 15 \qquad xyz = 15 \cdot 30 \cdot 45 = 2 \cdot 3^4 \cdot 5^3$

5. Find the area inside the **inner** loop of the limacon $r = \sqrt{3} - 2\cos\theta$.



a. $\frac{5}{3}\pi + \frac{1}{2}\sqrt{3}$

b. $\frac{5}{3}\pi + \frac{1}{2}\sqrt{3} - 6$

c. $6 - \frac{1}{2}\sqrt{3} - \frac{5}{3}\pi$

d. $\frac{5}{6}\pi - \frac{3}{2}\sqrt{3}$ Part Credit = 4 points

e. $\frac{3}{2}\sqrt{3} - \frac{5}{6}\pi$ Correct Choice

$r = 0$ when $\sqrt{3} - 2\cos\theta = 0$ or $\cos\theta = \frac{\sqrt{3}}{2}$ or $\theta = \pm\frac{\pi}{6}$

For $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$, we have $r < 0$. So the Jacobian is $J = -r$. (Part credit if missed.)

$$A = \int_{-\pi/6}^{\pi/6}\int_0^{\sqrt{3}-2\cos\theta} -r\,dr\,d\theta = -\int_{-\pi/6}^{\pi/6}\left[\frac{r^2}{2}\right]_{r=0}^{\sqrt{3}-2\cos\theta} d\theta = -\frac{1}{2}\int_{-\pi/6}^{\pi/6}\left(\sqrt{3} - 2\cos\theta\right)^2 d\theta$$

$$= \frac{1}{-2}\int_{-\pi/6}^{\pi/6}\left(3 - 4\sqrt{3}\cos\theta + 4\cos^2\theta\right) d\theta = -\frac{1}{2}\int_{-\pi/6}^{\pi/6}\left(3 - 4\sqrt{3}\cos\theta + 2 + 2\cos 2\theta\right) d\theta$$

$$= -\frac{1}{2}\left[5\theta - 4\sqrt{3}\sin\theta + \sin 2\theta\right]_{-\pi/6}^{\pi/6} = -\frac{2}{2}\left(5\frac{\pi}{6} - 4\sqrt{3}\sin\frac{\pi}{6} + \sin\frac{\pi}{3}\right)$$

$$= -\left(\frac{5\pi}{6} - 4\sqrt{3}\,\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{3}{2}\sqrt{3} - \frac{5}{6}\pi$$

6. Find the mass of the plate between $y = 2x$ and $y = x^2$ if the surface density is $\delta = xy$.

a. $\frac{8}{15}$
b. $\frac{16}{15}$
c. $\frac{32}{15}$
d. $\frac{8}{3}$ Correct Choice
e. $\frac{16}{3}$

$$M = \int\int \delta\, dA = \int_0^2 \int_{x^2}^{2x} xy\, dy\, dx = \int_0^2 x\left[\frac{y^2}{2}\right]_{x^2}^{2x} dx = \frac{1}{2}\int_0^2 x(4x^2 - x^4)\, dx = \frac{1}{2}\left[x^4 - \frac{x^6}{6}\right]_0^2$$
$$= \frac{1}{2}\left(16 - \frac{32}{3}\right) = \frac{8}{3}$$

7. Find the y-component of the center of mass of the plate between $y = 2x$ and $y = x^2$ if the surface density is $\delta = xy$.

a. $\frac{8}{15}$
b. $\frac{5}{12}$
c. $\frac{12}{5}$ Correct Choice
d. $\frac{15}{8}$
e. $\frac{32}{5}$

$$M_y = \int\int y\delta\, dA = \int_0^2 \int_{x^2}^{2x} xy^2\, dy\, dx = \int_0^2 x\left[\frac{y^3}{3}\right]_{x^2}^{2x} dx = \frac{1}{3}\int_0^2 x(8x^3 - x^6)\, dx$$
$$= \frac{1}{3}\left[8\frac{x^5}{5} - \frac{x^8}{8}\right]_0^2 = \frac{1}{3}\left(\frac{2^8}{5} - \frac{2^8}{8}\right) = \frac{2^8}{3}\left(\frac{8-5}{40}\right) = \frac{2^5}{5} = \frac{32}{5}$$
$$\bar{y} = \frac{M_y}{M} = \frac{32}{5}\frac{3}{8} = \frac{12}{5}$$

8. Compute $\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x + y, x + 2y, 2z)$ along the curve $\vec{r}(t) = (\sqrt{t}, t^2 + t^3, t^2 - t^3)$.
HINT: Find a scalar potential if possible.

a. 1
b. 3
c. 5
d. 7 Correct Choice
e. Cannot be computed because there is no scalar potential.

$$\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f) = \vec{F} = (2x + y, x + 2y, 2z) \qquad \Rightarrow \qquad f = x^2 + xy + y^2 + z^2$$

By the FTCC, $\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s} = \int_{(0,0,0)}^{(1,2,0)} \vec{\nabla} f \cdot d\vec{s} = f(1,2,0) - f(0,0,0) = 7$

9. Compute $\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (x,x,x)$ along the curve $\vec{r}(t) = (t, t^2 + t^3, t^2 - t^3)$.

HINT: Find a scalar potential if possible.

a. $\frac{11}{6}$ Correct Choice

b. $-\frac{5}{6}$

c. $\frac{5}{6}$

d. 3

e. Cannot be computed because there is no scalar potential.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & x & x \end{vmatrix} = (0,-1,1) \neq (0,0,0) \qquad \Rightarrow \qquad \text{No scalar potential.}$$

$$\vec{F}(\vec{r}(t)) = (t,t,t) \qquad \vec{v} = (1, 2t + 3t^2, 2t - 3t^2)$$

$$\vec{F} \cdot \vec{v} = t + t(2t + 3t^2) + t(2t - 3t^2) = t + 4t^2$$

$$\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F} \cdot \vec{v}\, dt = \int_0^1 (t + 4t^2)\, dt = \left[\frac{t^2}{2} + \frac{4t^3}{3} \right]_0^1 = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}$$

10. Compute the line integral $\oint \vec{F} \cdot d\vec{s}$

for $\vec{F} = (2y - 3xy^2, 3x - 3x^2 y)$

counterclockwise around the boundary

of the region shown at the right.

HINT: Use a Theorem.

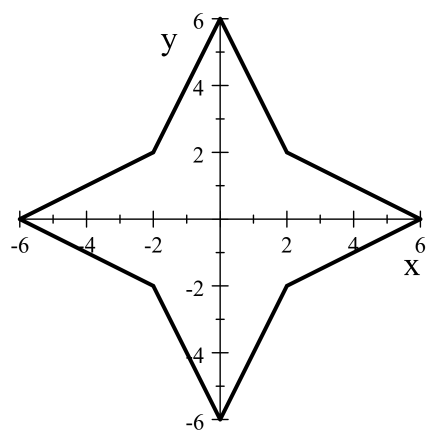
a. 3

b. 6

c. 12

d. 24

e. 48 Correct Choice



Apply Green's Theorem. $P = 2y - 3xy^2 \qquad Q = 3x - 3x^2 y$

$$\partial_x Q - \partial_y P = \partial_x(3x - 3x^2 y) - \partial_y(2y - 3xy^2) = (3 - 6xy) - (2 - 6xy) = 1$$

$$\oint \vec{F} \cdot d\vec{s} = \oint P\, dx + Q\, dy = \iint (\partial_x Q - \partial_y P)\, dx\, dy = \iint 1\, dx\, dy$$

$= \text{Area} = \text{Area of square} + 4 \cdot \text{Area of triangle} = (4 \cdot 4) + 4\left(\frac{1}{2} 4 \cdot 4\right) = 48$

11. Compute $\iint_Q \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for

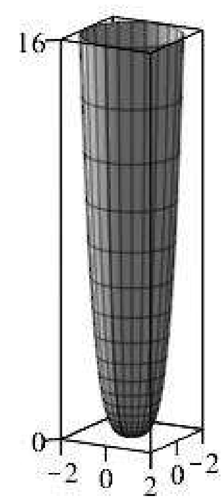
$\vec{F} = \left(x(z-16)^2 - yz^3, y(z-16)^2 + xz^3, x^4z - y^4z\right)$

over the quartic surface Q given by $z = (x^2+y^2)^2$

for $z \le 16$ oriented down and out, which may be

parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^4)$.

HINT: Use a Theorem. Be sure to check the orientation.



a. $-2^{18}\pi$

b. $-2^{16}\pi$

c. $-2^{15}\pi$ Correct Choice

d. $2^{15}\pi$ Part Credit = 4 points

e. $2^{18}\pi$

By Stokes' Theorem, $\iint_Q \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial Q} \vec{F} \cdot d\vec{s}$. The boundary of Q is given by $z = (x^2+y^2)^2 = 16$ which is the circle $x^2+y^2 = 4$ with $z = 16$. It may be parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 16)$. The velocity is $\vec{v} = (-2\sin\theta, 2\sin\theta, 0)$. This is counterclockwise as seen from above. We need clockwise. So we reverse the velocity to $\vec{v} = (2\sin\theta, -2\sin\theta, 0)$. On the circle, the vector field is

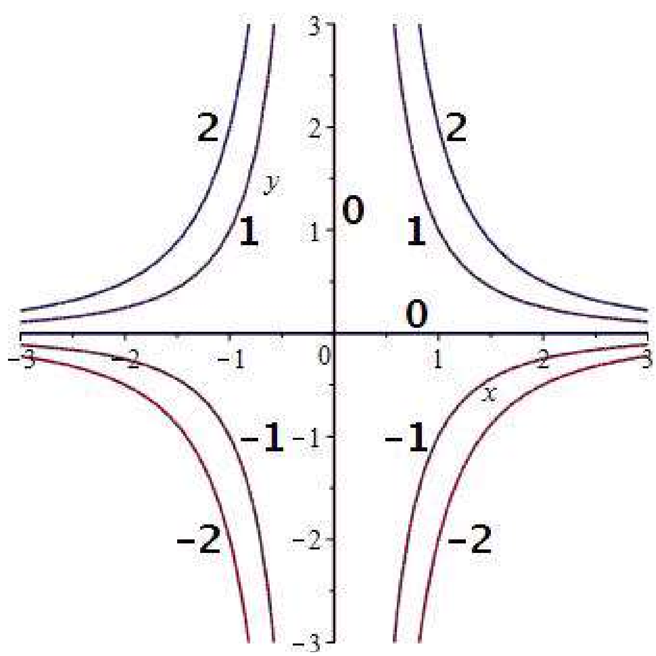
$$\begin{aligned}\vec{F} &= \left(x(z-16)^2 - yz^3, y(z-16)^2 + xz^3, x^4z - y^4z\right) \\ &= \left(2\cos\theta(0)^2 - 2\sin\theta 16^3, 2\sin\theta(0)^2 + 2\cos\theta 16^3, (2\cos\theta)^4 16 - (2\sin\theta)^4 16\right) \\ &= \left(-2\sin\theta 16^3, 2\cos\theta 16^3, (2\cos\theta)^4 16 - (2\sin\theta)^4 16\right)\end{aligned}$$

So $\vec{F} \cdot \vec{v} = -4\sin^2\theta 16^3 - 4\cos^2\theta 16^3 = -4 \cdot 16^3 = -2^{14}$

$$\oint_{\partial Q} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -2^{14}\, d\theta = -2^{15}\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (10 points) Roughly draw the contour plot for the function $f(x,y) = x^2y$. Include and label the level sets for $f = -2, -1, 0, 1, 2$. If there is more than one piece to a level set, label each piece.

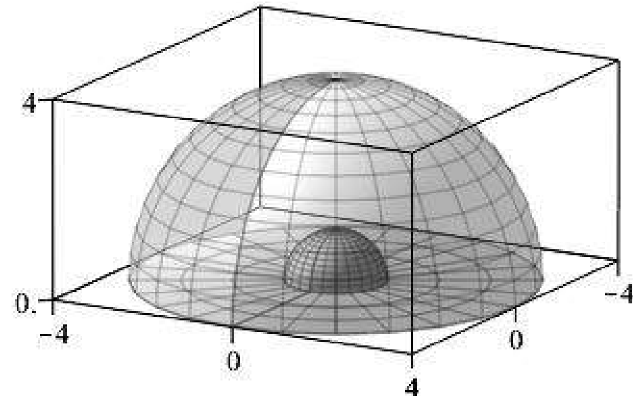


13. (30 points) Verify Gauss' Theorem

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$$

for the vector field $\vec{F} = (xz^2, yz^2, z^3)$ and the solid region, V, between the hemispheres $z = \sqrt{16 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$ for $z \geq 0$.

Use the following steps: Be sure to check orientations.



a. (7 pts) LHS:

$\vec{\nabla} \cdot \vec{F} = \partial_x(xz^2) + \partial_y(yz^2) + \partial_z(z^3) = 5z^2$

Name your coordinate system Spherical and evaluate

$\vec{\nabla} \cdot \vec{F}\Big|_{\vec{R}} = 5\rho^2 \cos^2\varphi$ $\qquad dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_1^4 5\rho^2 \cos^2\varphi \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \left[-\frac{\cos^3\varphi}{3} \right]_0^{\pi/2} \left[\rho^5 \right]_1^4$$

$$= 2\pi \left(\frac{1}{3} \right) (1023) = 682\pi$$

b. RHS: The boundary consists of 3 pieces.

i. (13 pts) Parametrize the Outer Hemisphere: $\vec{R}(\varphi, \theta) = (4\sin\varphi\cos\theta, 4\sin\varphi\sin\theta, 4\cos\varphi)$

Evaluate the vector field on the surface:

$\vec{F}\left(\vec{R}\right) = (xz^2, yz^2, z^3) = (4 \cdot 16\cos^2\varphi \sin\varphi \cos\theta, 4 \cdot 16\cos^2\varphi \sin\varphi \sin\theta, 64\cos^3\varphi)$

Find the normal:

$\vec{e}_\varphi = (4\cos\varphi\cos\theta, 4\cos\varphi\sin\theta, -4\sin\varphi)$

$\vec{e}_\theta = (-4\sin\varphi\sin\theta, 4\sin\varphi\cos\theta, 0)$

$\vec{N} = (16\sin^2\varphi\cos\theta, 16\sin^2\varphi\sin\theta, 16\sin\varphi\cos\varphi)$ correctly oriented outward.

Evaluate:

$$\vec{F} \cdot \vec{N} = 4 \cdot 16^2 \cos^2\varphi \sin^3\varphi \cos^2\theta + 4 \cdot 16^2 \cos^2\varphi \sin^3\varphi \sin^2\theta + 16 \cdot 64\cos^4\varphi \sin\varphi$$
$$= 4 \cdot 16^2 (\cos^2\varphi \sin^3\varphi + \cos^4\varphi \sin\varphi) = 4 \cdot 16^2 \cos^2\varphi \sin\varphi (\sin^2\varphi + \cos^2\varphi)$$
$$= 4 \cdot 16^2 \cos^2\varphi \sin\varphi$$

$$\iint_{\text{outer}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi/2} 4 \cdot 16^2 \cos^2\varphi \sin\varphi \, d\varphi \, d\theta = 2\pi 4 \cdot 16^2 \left[-\frac{\cos^3\varphi}{3} \right]_0^{\pi/2} = \frac{2048}{3}\pi$$

(continued)

ii. (3 pts) Parametrize the Inner Hemisphere: $\vec{R}(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

Evaluate the vector field on the surface:

$\vec{F}(\vec{R}) = (xz^2, yz^2, z^3) = (\cos^2\varphi\sin\varphi\cos\theta, \cos^2\varphi\sin\varphi\sin\theta, \cos^3\varphi)$

Find the normal:

$\vec{e}_\varphi = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$

$\vec{e}_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$

$\vec{N} = (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)$ oriented outward, need inward.

Reverse $\vec{N} = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$

Evaluate:

$\vec{F}\cdot\vec{N} = -\cos^2\varphi\sin^3\varphi\cos^2\theta - \cos^2\varphi\sin^3\varphi\sin^2\theta - \cos^4\varphi\sin\varphi = \cdots = -\cos^2\varphi\sin\varphi$

$$\iint_{\text{inner}} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} -\cos^2\varphi\sin\varphi\, d\varphi\, d\theta = -2\pi\left[-\frac{\cos^3\varphi}{3}\right]_0^{\pi/2} = -\frac{2}{3}\pi$$

iii. (3 pts) Parametrize the Base Ring: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$

Evaluate the vector field on the surface:

$\vec{F}(\vec{R}) = (xz^2, yz^2, z^3) = (0,0,0)$

Find the normal:

$\vec{e}_r = (\cos\theta, \sin\theta, 0)$ $\quad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$ You may skip this.

$\vec{N} = (0,0,r)$ oriented outward, need inward. Reverse $\vec{N} = (0,0,-r)$ You may skip this.

Evaluate:

$$\iint_{\text{ring}} \vec{F}\cdot d\vec{S} = \iint_{\text{ring}} \vec{F}\cdot\vec{N}\, dr\, d\theta = 0$$

iv. (2 pts) Total RHS:

$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_{\text{outer}} \vec{F}\cdot d\vec{S} + \iint_{\text{inner}} \vec{F}\cdot d\vec{S} + \iint_{\text{ring}} \vec{F}\cdot d\vec{S} = \frac{2048}{3}\pi - \frac{2}{3}\pi + 0 = 682\pi$$

c. (2 pts) Comparison of LHS and RHS: They are the same.

14. (12 points) Find all points on the paraboloid $z = x^2 + y^2$ where the normal line passes throught the point $P = (0,0,36)$.

HINT: The normal vector at $X = (x,y,z)$ must be parallel to the vector $\overrightarrow{XP}$.

Let $g = z - x^2 - y^2$. Then the paraboloid is the level set $g = 0$, its normal at X is the gradient at X:

$\vec{\nabla} g = (-2x, -2y, 1)$

This is parallel to $\overrightarrow{XP} = P - X = (0 - x, 0 - y, 36 - z)$ iff $\vec{\nabla} g = \lambda \overrightarrow{XP}$. So we need to solve:

$-2x = -\lambda x \qquad -2y = -\lambda y \qquad 1 = \lambda(36 - z) \qquad z = x^2 + y^2$

The 1^{st} eq. says $\lambda = 2$ or $x = 0$. The 2^{nd} eq. says $\lambda = 2$ or $y = 0$.

Case 1: $\lambda = 2$:

The 3^{rd} eq. says $1 = 2(36 - z)$ or $z = 36 - \frac{1}{2} = \frac{71}{2}$

The constraint says $x^2 + y^2 = \frac{71}{2}$.

So the solution is any point $X = (x,y,z)$ with $z = \frac{71}{2}$ and x and y on the circle $x^2 + y^2 = \frac{71}{2}$.

Case 2: $\lambda \neq 2$:

Then $x = y = 0$ and the constraint says $z = x^2 + y^2 = 0$.

(The 3^{rd} eq. says $1 = \lambda(36 - z) = \lambda(36)$ or $\lambda = \frac{1}{36}$ which we don't care about.)

So the solution is the single point $(0,0,0)$.

Final Solution:

Any point on the circle $x^2 + y^2 = \frac{71}{2}$ with $z = \frac{71}{2}$ or the origin $(0,0,0)$.