MATH 304
Linear Algebra

Lecture 12:
Rank and nullity of a matrix.
Definition. Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a basis.

Equivalently, a subset $S \subset V$ is a basis for $V$ if any vector $v \in V$ is uniquely represented as a linear combination

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k,$$

where $v_1, \ldots, v_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$. 
**Theorem 1**  Any vector space has a basis.

**Theorem 2**  If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

**Definition.** The **dimension** of a vector space $V$, denoted $\dim V$, is the number of elements in any of its bases.
Examples.  

• \( \dim \mathbb{R}^n = n \)

• \( \mathcal{M}_{2,2}(\mathbb{R}) \): the space of \( 2 \times 2 \) matrices
  \( \dim \mathcal{M}_{2,2}(\mathbb{R}) = 4 \)

• \( \mathcal{M}_{m,n}(\mathbb{R}) \): the space of \( m \times n \) matrices
  \( \dim \mathcal{M}_{m,n}(\mathbb{R}) = mn \)

• \( \mathcal{P}_n \): polynomials of degree less than \( n \)
  \( \dim \mathcal{P}_n = n \)

• \( \mathcal{P} \): the space of all polynomials
  \( \dim \mathcal{P} = \infty \)

• \( \{0\} \): the trivial vector space
  \( \dim \{0\} = 0 \)
**Row space of a matrix**

*Definition.* The **row space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^n$ spanned by rows of $A$.

The dimension of the row space is called the **rank** of the matrix $A$.

**Theorem 1** The rank of a matrix $A$ is the maximal number of linearly independent rows in $A$.

**Theorem 2** Elementary row operations do not change the row space of a matrix.

**Theorem 3** If a matrix $A$ is in row echelon form, then the nonzero rows of $A$ are linearly independent.

**Corollary** The rank of a matrix is equal to the number of nonzero rows in its row echelon form.
Theorem  Elementary row operations do not change the row space of a matrix.

Proof:  Suppose that $A$ and $B$ are $m \times n$ matrices such that $B$ is obtained from $A$ by an elementary row operation. Let $a_1, \ldots, a_m$ be the rows of $A$ and $b_1, \ldots, b_m$ be the rows of $B$. We have to show that $\text{Span}(a_1, \ldots, a_m) = \text{Span}(b_1, \ldots, b_m)$.

Observe that any row $b_i$ of $B$ belongs to $\text{Span}(a_1, \ldots, a_m)$. Indeed, either $b_i = a_j$ for some $1 \leq j \leq m$, or $b_i = ra_i$ for some scalar $r \neq 0$, or $b_i = a_i + ra_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\text{Span}(b_1, \ldots, b_m) \subset \text{Span}(a_1, \ldots, a_m)$.

Now the matrix $A$ can also be obtained from $B$ by an elementary row operation. By the above,

$$\text{Span}(a_1, \ldots, a_m) \subset \text{Span}(b_1, \ldots, b_m).$$
Problem. Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Elementary row operations do not change the row space. Let us convert $A$ to row echelon form:

\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Vectors \((1, 1, 0), (0, 1, 1),\) and \((0, 0, 1)\) form a basis for the row space of \(A\). Thus the rank of \(A\) is 3.

It follows that the row space of \(A\) is the entire space \(\mathbb{R}^3\).
Problem. Find a basis for the vector space $V$ spanned by vectors $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$, and $w_4 = (1, 1, 1)$.

The vector space $V$ is the row space of a matrix

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 3 & 1 \\
1 & 1 & 1
\end{pmatrix}.
$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for $V$. 
Column space of a matrix

Definition. The column space of an \( m \times n \) matrix \( A \) is the subspace of \( \mathbb{R}^m \) spanned by columns of \( A \).

Theorem 1  The column space of a matrix \( A \) coincides with the row space of the transpose matrix \( A^T \).

Theorem 2  Elementary column operations do not change the column space of a matrix.

Theorem 3  Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

Theorem 4  For any matrix, the row space and the column space have the same dimension.
Problem. Find a basis for the column space of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

The column space of \( A \) coincides with the row space of \( A^T \). To find a basis, we convert \( A^T \) to row echelon form:

\[ A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Vectors \((1, 0, 2, 1), (0, 1, 1, 0), \) and \((0, 0, 0, 1)\) form a basis for the column space of \( A \).
Problem. Find a basis for the column space of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

Alternative solution: We already know from a previous problem that the rank of \( A \) is 3. It follows that the columns of \( A \) are linearly independent. Therefore these columns form a basis for the column space.
Nullspace of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

**Definition.** The **nullspace** of the matrix $A$, denoted $N(A)$, is the set of all $n$-dimensional column vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with $A$ as the coefficient matrix).
Let $A = (a_{ij})$ be an $m \times n$ matrix.

**Theorem** The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

**Proof:** We have to show that $N(A)$ is nonempty, closed under addition, and closed under scaling.

First of all, $A \mathbf{0} = \mathbf{0} \implies \mathbf{0} \in N(A) \implies N(A)$ is not empty.

Secondly, if $\mathbf{x}, \mathbf{y} \in N(A)$, i.e., if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x} + \mathbf{y} \in N(A)$.

Thirdly, if $\mathbf{x} \in N(A)$, i.e., if $A\mathbf{x} = \mathbf{0}$, then for any $r \in \mathbb{R}$ one has $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0} \implies r\mathbf{x} \in N(A)$.

**Definition.** The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix $A$. 
Problem. Find the nullity of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}. \]

Elementary row operations do not change the nullspace. Let us convert \( A \) to reduced row echelon form:

\[
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}
\]

\[
\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}
\]

General element of \( N(A) \):

\[
(x_1, x_2, x_3, x_4) = (t + 2s, -2t - 3s, t, s) = t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}.
\]

Vectors \((1, -2, 1, 0)\) and \((2, -3, 0, 1)\) form a basis for \( N(A) \). Thus the nullity of the matrix \( A \) is 2.
Theorem  The rank of a matrix $A$ plus the nullity of $A$ equals the number of columns in $A$.

*Sketch of the proof:* The rank of $A$ equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of $A$ equals the number of free variables in the corresponding system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank + nullity is the number of all columns in the matrix $A$. 
Problem. Find the nullity of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}. \]

Alternative solution: Clearly, the rows of \( A \) are linearly independent. Therefore the rank of \( A \) is 2. Since

\[
(\text{rank of } A) + (\text{nullity of } A) = 4,
\]

it follows that the nullity of \( A \) is 2.