Lecture 23:
Eigenvalues and eigenvectors of a linear operator.
Basis of eigenvectors.
Eigenvalues and eigenvectors of a matrix

*Definition.* Let $A$ be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix $A$ if $A \mathbf{v} = \lambda \mathbf{v}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{v}$ is called an **eigenvector** of $A$ belonging to (or associated with) the eigenvalue $\lambda$.

If $\lambda$ is an eigenvalue of $A$ then the nullspace $N(A - \lambda I)$, which is nontrivial, is called the **eigenspace** of $A$ corresponding to $\lambda$. The eigenspace consists of all eigenvectors belonging to the eigenvalue $\lambda$ plus the zero vector.
**Characteristic equation**

*Definition.* Given a square matrix $A$, the equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of $A$.

Eigenvalues $\lambda$ of $A$ are roots of the characteristic equation.

If $A$ is an $n \times n$ matrix then $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$. It is called the **characteristic polynomial** of $A$.

**Theorem** Any $n \times n$ matrix has at most $n$ eigenvalues.
Eigenvalues and eigenvectors of an operator

Definition. Let $V$ be a vector space and $L : V \rightarrow V$ be a linear operator. A number $\lambda$ is called an eigenvalue of the operator $L$ if $L(v) = \lambda v$ for a nonzero vector $v \in V$. The vector $v$ is called an eigenvector of $L$ associated with the eigenvalue $\lambda$. (If $V$ is a functional vector space then eigenvectors are usually called eigenfunctions.)

If $V = \mathbb{R}^n$ then the linear operator $L$ is given by $L(x) = Ax$, where $A$ is an $n \times n$ matrix (and $x$ is regarded a column vector). In this case, eigenvalues and eigenvectors of the operator $L$ are precisely eigenvalues and eigenvectors of the matrix $A$. 
Eigenspaces

Let $L : V \rightarrow V$ be a linear operator.

For any $\lambda \in \mathbb{R}$, let $V_\lambda$ denotes the set of all solutions of the equation $L(x) = \lambda x$.

Then $V_\lambda$ is a subspace of $V$ since $V_\lambda$ is the kernel of a linear operator given by $x \mapsto L(x) - \lambda x$.

$V_\lambda$ minus the zero vector is the set of all eigenvectors of $L$ associated with the eigenvalue $\lambda$.

In particular, $\lambda \in \mathbb{R}$ is an eigenvalue of $L$ if and only if $V_\lambda \neq \{0\}$.

If $V_\lambda \neq \{0\}$ then it is called the eigenspace of $L$ corresponding to the eigenvalue $\lambda$. 
Example. \( V = C^\infty(\mathbb{R}) \), \( D : V \to V \), \( Df = f' \).

A function \( f \in C^\infty(\mathbb{R}) \) is an eigenfunction of the operator \( D \) belonging to an eigenvalue \( \lambda \) if \( f'(x) = \lambda f(x) \) for all \( x \in \mathbb{R} \).

It follows that \( f(x) = ce^{\lambda x} \), where \( c \) is a nonzero constant.

Thus each \( \lambda \in \mathbb{R} \) is an eigenvalue of \( D \). The corresponding eigenspace is spanned by \( e^{\lambda x} \).
Example. \( V = C^\infty(\mathbb{R}) \), \( L : V \to V \), \( Lf = f'' \).

\( Lf = \lambda f \iff f''(x) - \lambda f(x) = 0 \) for all \( x \in \mathbb{R} \).

It follows that each \( \lambda \in \mathbb{R} \) is an eigenvalue of \( L \) and the corresponding eigenspace \( V_\lambda \) is two-dimensional. Note that \( L = D^2 \), hence \( Df = \mu f \implies Lf = \mu^2 f \).

If \( \lambda > 0 \) then \( V_\lambda = \text{Span}(e^{\mu x}, e^{-\mu x}) \), where \( \mu = \sqrt{\lambda} \).

If \( \lambda < 0 \) then \( V_\lambda = \text{Span}(\sin(\mu x), \cos(\mu x)) \), where \( \mu = \sqrt{-\lambda} \).

If \( \lambda = 0 \) then \( V_\lambda = \text{Span}(1, x) \).
Suppose \( L : V \to V \) is a linear operator on a finite-dimensional vector space \( V \).

Let \( u_1, u_2, \ldots, u_n \) be a basis for \( V \) and \( g : V \to \mathbb{R}^n \) be the corresponding coordinate mapping. Let \( A \) be the matrix of \( L \) with respect to this basis. Then

\[
L(v) = \lambda v \iff A g(v) = \lambda g(v).
\]

Hence the eigenvalues of \( L \) coincide with those of the matrix \( A \). Moreover, the associated eigenvectors of \( A \) are coordinates of the eigenvectors of \( L \).

**Definition.** The characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \) of the matrix \( A \) is called the **characteristic polynomial** of the operator \( L \).

Then eigenvalues of \( L \) are roots of its characteristic polynomial.
Theorem. The characteristic polynomial of the operator $L$ is well defined. That is, it does not depend on the choice of a basis.

Proof: Let $B$ be the matrix of $L$ with respect to a different basis $v_1, v_2, \ldots, v_n$. Then $A = UBU^{-1}$, where $U$ is the transition matrix from the basis $v_1, \ldots, v_n$ to $u_1, \ldots, u_n$. We have to show that $\det(A - \lambda I) = \det(B - \lambda I)$ for all $\lambda \in \mathbb{R}$. We obtain

$$\det(A - \lambda I) = \det(UBU^{-1} - \lambda I)$$

$$= \det(UBU^{-1} - U(\lambda I)U^{-1}) = \det(U(B - \lambda I)U^{-1})$$

$$= \det(U) \det(B - \lambda I) \det(U^{-1}) = \det(B - \lambda I).$$
Basis of eigenvectors

Let $V$ be a finite-dimensional vector space and $L : V \rightarrow V$ be a linear operator. Let $v_1, v_2, \ldots, v_n$ be a basis for $V$ and $A$ be the matrix of the operator $L$ with respect to this basis.

**Theorem** The matrix $A$ is diagonal if and only if vectors $v_1, v_2, \ldots, v_n$ are eigenvectors of $L$.

If this is the case, then the diagonal entries of the matrix $A$ are the corresponding eigenvalues of $L$.

$$ L(v_i) = \lambda_i v_i \iff A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} $$
**How to find a basis of eigenvectors**

**Theorem**  If \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are eigenvectors of a linear operator \( L \) associated with distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \), then \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are linearly independent.

**Corollary 1**  Suppose \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are all eigenvalues of a linear operator \( L : V \rightarrow V \). For any \( 1 \leq i \leq k \), let \( S_i \) be a basis for the eigenspace associated to the eigenvalue \( \lambda_i \). Then these bases are disjoint and the union \( S = S_1 \cup S_2 \cup \cdots \cup S_k \) is a linearly independent set.

Moreover, if the vector space \( V \) admits a basis consisting of eigenvectors of \( L \), then \( S \) is such a basis.

**Corollary 2**  Let \( A \) be an \( n \times n \) matrix such that the characteristic equation \( \det(A - \lambda I) = 0 \) has \( n \) distinct roots. Then (i) there is a basis for \( \mathbb{R}^n \) consisting of eigenvectors of \( A \); (ii) all eigenspaces of \( A \) are one-dimensional.
**Theorem 1**  Let $L$ be a linear operator on a finite-dimensional vector space $V$. Then the following conditions are equivalent:
- the matrix of $L$ with respect to some basis is diagonal;
- there exists a basis for $V$ formed by eigenvectors of $L$.

The operator $L$ is **diagonalizable** if it satisfies these conditions.

**Theorem 2**  Let $A$ be an $n \times n$ matrix. Then the following conditions are equivalent:
- $A$ is the matrix of a diagonalizable operator;
- $A$ is similar to a diagonal matrix, i.e., it is represented as $A = UBU^{-1}$, where the matrix $B$ is diagonal;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of $A$.

The matrix $A$ is **diagonalizable** if it satisfies these conditions.
Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix $A$ has two eigenvalues: 1 and 3.
- The eigenspace of $A$ associated with the eigenvalue 1 is the line spanned by $v_1 = (-1, 1)$.
- The eigenspace of $A$ associated with the eigenvalue 3 is the line spanned by $v_2 = (1, 1)$.
- Eigenvectors $v_1$ and $v_2$ form a basis for $\mathbb{R}^2$.

Thus the matrix $A$ is diagonalizable. Namely,

$A = UBU^{-1}$, where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

Notice that $U$ is the transition matrix from the basis $v_1, v_2$ to the standard basis.
Example. \[ A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \]

- The matrix \( A \) has two eigenvalues: 0 and 2.
- The eigenspace for 0 is one-dimensional; it has a basis \( S_1 = \{\mathbf{v}_1\} \), where \( \mathbf{v}_1 = (-1, 1, 0) \).
- The eigenspace for 2 is two-dimensional; it has a basis \( S_2 = \{\mathbf{v}_2, \mathbf{v}_3\} \), where \( \mathbf{v}_2 = (1, 1, 0) \), \( \mathbf{v}_3 = (-1, 0, 1) \).
- The union \( S_1 \cup S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a linearly independent set, hence it is a basis for \( \mathbb{R}^3 \).

Thus the matrix \( A \) is diagonalizable. Namely, \( A = UBU^{-1} \), where

\[
B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
There are **two obstructions** to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

**Example 1.**  \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

\[ \det(A - \lambda I) = (\lambda - 1)^2. \]

Hence \( \lambda = 1 \) is the only eigenvalue. The associated eigenspace is the line \( t(1, 0) \).

**Example 2.**  \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

\[ \det(A - \lambda I) = \lambda^2 + 1. \]

\( \implies \) no real eigenvalues or eigenvectors

(However there are complex eigenvalues/eigenvectors.)