Lecture 20:
Topology of the real line:
open and closed sets.
Classification of points

Let $E \subset \mathbb{R}$ be a subset of the real line and $x \in \mathbb{R}$ be a point. Recall that for any $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon)$ is called the $\varepsilon$-neighborhood of the point $x$ as it consists of all points at distance less than $\varepsilon$ from $x$.

Definition. The point $x$ is called an interior point of the set $E$ if for some $\varepsilon > 0$ the entire $\varepsilon$-neighborhood $(x - \varepsilon, x + \varepsilon)$ is contained in $E$. The point $x$ is called an exterior point of $E$ if for some $\varepsilon > 0$ the $\varepsilon$-neighborhood $(x - \varepsilon, x + \varepsilon)$ is disjoint from $E$. The point $x$ is called a boundary point of $E$ if for any $\varepsilon > 0$ the $\varepsilon$-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains both a point in $E$ and another point not in $E$.

Remark. Every interior point of the set $E$ must belong to $E$. Every exterior point of $E$ must not belong to $E$. Any particular boundary point may or may not be in $E$. 
Examples

• $E = (a, b)$, an open interval.
  The interior points are points in $(a, b)$. The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are $a$ and $b$.

• $E = [a, b]$, a closed interval.
  The interior points are points in $(a, b)$. The exterior points are points in $(-\infty, a) \cup (b, +\infty)$. The boundary points are $a$ and $b$.

• $E = [0, 1) \cup [2, 3)$.
  The interior points are points in $(0, 1) \cup (2, 3)$. The exterior points are points in $(-\infty, 0) \cup (1, 2) \cup (3, +\infty)$. The boundary points are $0, 1, 2$ and $3$. 
Examples

- \( E = \mathbb{R} \), the entire real line.
  Every point is interior. There are no boundary or exterior points.

- \( E = \emptyset \), the empty set.
  Every point is exterior. There are no interior or boundary points.

- \( E = \mathbb{Q} \), the rational numbers.
  Every open interval \((a, b)\) contains a rational number (the rational numbers are dense). Also, \((a, b)\) contains an irrational number (since \(\mathbb{Q}\) is countable while the interval is not). Therefore every point of \(\mathbb{R}\) is a boundary point for \(\mathbb{Q}\). There are no interior or exterior points.
Examples

- \( E = \mathbb{N} \), the natural numbers.

  Every natural number is a boundary point. Any non-natural number is an exterior point since the complement \( \mathbb{R} \setminus \mathbb{N} \) is a union of open intervals: \( \mathbb{R} \setminus \mathbb{N} = (−\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots \). There are no interior points.

- \( E = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\} \), a monotonic sequence.

  The boundary points are all points of \( E \) and 0 (the limit of the sequence). All the other points are exterior. There are no interior points.
Let $E \subset \mathbb{R}$ be a subset of the real line.

Definition. The set of all interior points of $E$ is called the **interior** of $E$ and denoted $\text{int}(E)$. The set of all boundary points of $E$ is called the **boundary** of $E$ and denoted $\partial E$. The set of all exterior points of $E$ is called the **exterior** of $E$.

**Proposition 1** The exterior of the set $E$ coincides with $\text{int}(\mathbb{R} \setminus E)$, the interior of its complement.

**Proposition 2** The boundary of the set $E$ coincides with the boundary of its complement: $\partial E = \partial (\mathbb{R} \setminus E)$.

**Proposition 3** The real line $\mathbb{R}$ is the disjoint union of three sets: $\mathbb{R} = \text{int}(E) \cup \partial E \cup \text{int}(\mathbb{R} \setminus E)$.

**Proposition 4** $\text{int}(E) \subset E \subset \text{int}(E) \cup \partial E$. 
Limit points of a set

Let $E \subset \mathbb{R}$ be a subset of the real line.

**Definition.** A point $x \in \mathbb{R}$ is called a **limit point** of the set $E$ if there exists a sequence $x_1, x_2, x_3, \ldots$ such that each $x_n$ belongs to $E$ and $x_n \to x$ as $n \to \infty$.

**Remark.** Elements of the sequence $\{x_n\}$ need not be distinct. In particular, every point $x \in E$ is a limit point of $E$, as the limit of a constant sequence $x, x, x, \ldots$

**Theorem**  A point $x \in \mathbb{R}$ is a limit point of a set $E \subset \mathbb{R}$ if and only if for any $\varepsilon > 0$ the $\varepsilon$-neighborhood $(x - \varepsilon, x + \varepsilon)$ contains at least one element of $E$.

**Corollary** For any set $E \subset \mathbb{R}$, the set of all limit points of $E$ is $\text{int}(E) \cup \partial E$. 
Another classification of points

Let \( E \subset \mathbb{R} \) be a subset of the real line and \( x \in \mathbb{R} \) be a point.

**Definition.** The point \( x \) is called an **accumulation point** of the set \( E \) if for any \( \varepsilon > 0 \) the \( \varepsilon \)-neighborhood \((x - \varepsilon, x + \varepsilon)\) contains infinitely many elements of \( E \). The point \( x \) is called an **isolated point** of \( E \) if \((x - \varepsilon, x + \varepsilon) \cap E = \{x\}\) for some \( \varepsilon > 0 \).

**Theorem** A point \( x \in \mathbb{R} \) is an accumulation point of a set \( E \subset \mathbb{R} \) if and only if for any \( \varepsilon > 0 \) the \( \varepsilon \)-neighborhood \((x - \varepsilon, x + \varepsilon)\) contains at least one element of \( E \) different from \( x \).

**Corollary** For any set \( E \subset \mathbb{R} \), the set of all limit points of \( E \) is a disjoint union of the set of its accumulation points and the set of its isolated points.
Examples

• \( E = \mathbb{N} \), the natural numbers.

Every natural number is an isolated point. There are no accumulation points.

\[
E = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}, \text{ a monotonic sequence.}
\]

Every element of \( E \) is an isolated point. The only accumulation point is 0.

• \( E = \mathbb{Q} \), the rational numbers.

Every point of \( \mathbb{R} \) is an accumulation point for \( \mathbb{Q} \). There are no isolated points.
Open and closed sets

Definition. A subset $E \subset \mathbb{R}$ of the real line is called open if every point of $E$ is an interior point. The subset $E$ is called closed if it contains all of its limit points (or, equivalently, if it contains all of its boundary points).

Properties of open and closed sets.

- Any open interval $(a, b)$ is an open set.
- Any closed interval $[a, b]$ is a closed set.
- If a set $E$ is open then the complement $\mathbb{R} \setminus E$ is closed.
- If a set $E$ is closed then the complement $\mathbb{R} \setminus E$ is open.
- The empty set and the entire real line $\mathbb{R}$ are both closed and open (in fact, these are the only sets with this property).
- Intersection of two open sets is also open.
- Union of any collection of open sets is also open.
- Union of two closed sets is also closed.
- Intersection of any collection of closed sets is also closed.