Lecture 21:
Open and closed sets (continued).
Compact sets.
Open and closed sets

Definition. A subset $E \subset \mathbb{R}$ of the real line is called **open** if every point of $E$ is an interior point. The subset $E$ is called **closed** if it contains all of its limit points (or, equivalently, if it contains all of its boundary points).

Properties of open and closed sets.

- Any open interval $(a, b)$ is an open set.
- Any closed interval $[a, b]$ is a closed set.
- If a set $E$ is open then the complement $\mathbb{R} \setminus E$ is closed.
- If a set $E$ is closed then the complement $\mathbb{R} \setminus E$ is open.
- The empty set and the entire real line $\mathbb{R}$ are both closed and open (in fact, these are the only sets with this property).
- Intersection of finitely many open sets is also open.
- Union of any collection of open sets is also open.
- Union of finitely many closed sets is also closed.
- Intersection of any collection of closed sets is also closed.
Interior, boundary, and closure

Recall that the set of all interior points of a set $E \subset \mathbb{R}$ is called the **interior** of $E$ and denoted $\text{int}(E)$. The set of all boundary points of $E$ is called the **boundary** of $E$ and denoted $\partial E$.

**Definition.** The set of all limit points of the set $E$, which is $\text{int}(E) \cup \partial E$, is called the **closure** of $E$ and denoted $\overline{E}$.

- The interior $\text{int}(E)$ is always an open set. In other words, $\text{int}(\text{int}(E)) = \text{int}(E)$.
- The interior $\text{int}(E)$ is the largest open subset of the set $E$.
- The closure $\overline{E}$ is always a closed set. In other words, $\overline{\overline{E}} = \overline{E}$.
- The closure $\overline{E}$ is the smallest closed set that contains $E$.
- The boundary $\partial E$ is always a closed set.
- If the set $E$ is closed then the boundary $\partial E$ has no interior points, that is, $\partial(\partial E) = \partial E$. In general, $\partial(\partial E) = \partial \overline{E} \subset \partial E$. 


**Proposition 1** Suppose \(x\) is an interior point of a set \(E \subset \mathbb{R}\). Then either \((x, \infty) \subset E\) or there exists a boundary point \(x_+ \in \partial E\) such that \(x < x_+\) and \((x, x_+) \subset E\).

*Proof:* Let \(S = \{y > x \mid (x, y) \subset E\}\). The set \(S\) is not empty since \(x\) is an interior point of \(E\). If \((x, \infty)\) is not contained in \(E\) then the set \(E\) is bounded above (since any \(z > x\) not in \(E\) is an upper bound for \(S\)). Therefore \(x_+ = \sup S\) is a finite number. Clearly, \(x_+ > x\). Any \(z \in (x, x_+)\) is not an upper bound for \(S\); hence \(y > z\) for some \(y \in S\). Then \(z \in (x, y) \subset E\). Thus \((x, x_+) \subset E\). At the same time, for any \(y > x_+\) the interval \((x, y)\) is not contained in \(E\). Then \([x_+, y)\) is not contained in \(E\) as well. It follows that \(x_+\) is a boundary point of \(E\).

**Proposition 2** Suppose \(x\) is an interior point of a set \(E \subset \mathbb{R}\). Then either \((-\infty, x) \subset E\) or there exists a boundary point \(x_- \in \partial E\) such that \(x_- < x\) and \((x_-, x) \subset E\).
Connectedness of the real line

**Corollary**  Between any interior point and any exterior point of a set $E \subset \mathbb{R}$, there is always a boundary point of $E$.

**Theorem**  The empty set and the entire real line are the only subsets of $\mathbb{R}$ that are both open and closed.

*Proof:* Suppose that a set $E \subset \mathbb{R}$ is both open and closed. Then it has no boundary points. By the corollary, either all points of $\mathbb{R}$ are interior points of $E$ or else all points of $\mathbb{R}$ are exterior points of $E$. In the former case, $E = \mathbb{R}$. In the latter case, $E$ is the empty set.
General open sets

**Theorem** Any nonempty open set \( E \subset \mathbb{R} \) decomposes as a union of a finite or infinite sequence of disjoint open intervals: 
\[ E = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \] Moreover, the intervals \((a_i, b_i)\) are determined uniquely up to rearranging them.

An open interval \((a, b) \subset E\) is called a **maximal subinterval** of \(E\) if there is no other interval \((c, d)\) such that \((a, b) \subset (c, d) \subset E\). The theorem is derived from Propositions 1 and 2 through a series of lemmas.

**Lemma 1** Any point of \(E\) is contained in a maximal subinterval.

**Lemma 2** Finite endpoints of a maximal subinterval do not belong to \(E\) (i.e., they are boundary points of \(E\)).

**Lemma 3** Distinct maximal subintervals are disjoint.

**Lemma 4** There are at most countably many maximal subintervals. (*Hint:* choose a rational point in each interval.)
Compact sets

**Theorem (Bolzano-Weierstrass)**  Any bounded sequence of real numbers has a convergent subsequence.

**Corollary**  Any sequence of points in a closed interval \([a, b]\) has a subsequence converging to some point in \([a, b]\).

*Definition.* Suppose that a set \(E \subset \mathbb{R}\) has the **Bolzano-Weierstrass property**: any sequence of points from \(E\) has a subsequence converging to some point in \(E\). Then the set \(E\) is called **compact** (or **sequentially compact**).

**Theorem**  A set \(E \subset \mathbb{R}\) is compact if and only if it is closed and bounded.

*Proof ("if"):* Suppose \(E\) is closed and bounded. Then any sequence \(\{x_n\}\) of points from \(E\) is bounded. Hence it has a convergent subsequence \(\{x_{n_k}\}\). The limit of the subsequence is a limit point of \(E\) and so it is in \(E\). Thus \(E\) is compact.