Lecture 18:
Sign of a permutation.
Permutations

Let $X$ be a finite set. A permutation of $X$ is a bijection from $X$ to itself. The set of all permutations of $\{1, 2, \ldots, n\}$ is called the symmetric group on $n$ symbols and denoted $S(n)$.

**Theorem** Any permutation can be expressed as a product of disjoint cycles. This cycle decomposition is unique up to rearrangement of the cycles involved.

**Theorem** Let $\pi$ be a permutation. Then there is a positive integer $m$ such that $\pi^m = \text{id}$.

The order of a permutation $\pi$, denoted $o(\pi)$, is defined as the smallest positive integer $m$ such that $\pi^m = \text{id}$.

**Theorem** Let $\pi \in S(n)$ and suppose that $\pi = \sigma_1 \sigma_2 \ldots \sigma_k$ is a decomposition of $\pi$ as a product of disjoint cycles. Then the order of $\pi$ is the least common multiple of the lengths of cycles $\sigma_1, \ldots, \sigma_k$. 
Sign of a permutation

Theorem 1 (i) Any permutation is a product of transpositions.
(ii) If $\pi = \tau_1 \tau_2 \ldots \tau_n = \tau'_1 \tau'_2 \ldots \tau'_m$, where $\tau_i, \tau'_j$ are transpositions, then the numbers $n$ and $m$ are of the same parity.

A permutation $\pi$ is called even if it is a product of an even number of transpositions, and odd if it is a product of an odd number of transpositions.

The sign $\text{sgn}(\pi)$ of the permutation $\pi$ is defined to be +1 if $\pi$ is even, and −1 if $\pi$ is odd.

Theorem 2 (i) $\text{sgn}(\pi\sigma) = \text{sgn}(\pi) \text{sgn}(\sigma)$ for any $\pi, \sigma \in S(n)$.
(ii) $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$ for any $\pi \in S(n)$.
(iii) $\text{sgn}(\text{id}) = 1$.
(iv) $\text{sgn}(\tau) = -1$ for any transposition $\tau$.
(v) $\text{sgn}(\sigma) = (-1)^{r-1}$ for any cycle $\sigma$ of length $r$. 
Let $\pi \in S(n)$ and $i, j$ be integers, $1 \leq i < j \leq n$. We say that the permutation $\pi$ preserves order of the pair $(i, j)$ if $\pi(i) < \pi(j)$. Otherwise $\pi$ makes an inversion. Denote by $N(\pi)$ the number of inversions made by the permutation $\pi$.

**Lemma 1** Let $\tau, \pi \in S(n)$ and suppose that $\tau$ is an adjacent transposition, $\tau = (k \ k+1)$. Then $|N(\tau \pi) - N(\pi)| = 1$.

*Proof:* For every pair $(i, j)$, $1 \leq i < j \leq n$, let us compare the order of pairs $\pi(i), \pi(j)$ and $\tau \pi(i), \tau \pi(j)$. We observe that the order differs exactly for one pair, when $\{\pi(i), \pi(j)\} = \{k, k+1\}$. The lemma follows.

**Lemma 2** Let $\pi \in S(n)$ and $\tau_1, \tau_2, \ldots, \tau_k$ be adjacent transpositions. Then (i) for any $\pi \in S(n)$ the numbers $k$ and $N(\tau_1 \tau_2 \ldots \tau_k \pi) - N(\pi)$ are of the same parity, (ii) the numbers $k$ and $N(\tau_1 \tau_2 \ldots \tau_k)$ are of the same parity.

*Sketch of the proof:* (i) follows from Lemma 1 by induction on $k$. (ii) is a particular case of part (i), when $\pi = \text{id}$. 
**Lemma 3 (i)** Any cycle of length $r$ is a product of $r-1$ transpositions. (ii) Any transposition is a product of an odd number of adjacent transpositions.

*Proof: (i) $\left(x_1 \ x_2 \ldots \ x_r\right) = \left(x_1 \ x_2\right)\left(x_2 \ x_3\right)\left(x_3 \ x_4\right)\ldots\left(x_{r-1} \ x_r\right)$. (ii) $(k \ k+r) = \sigma^{-1}(k \ k+1)\sigma$, where $\sigma = (k+1 \ k+2 \ldots \ k+r)$. By the above, $\sigma = (k+1 \ k+2)(k+2 \ k+3)\ldots(k+r-1 \ k+r)$ and $\sigma^{-1} = (k+r \ k+r-1)\ldots(k+3 \ k+2)(k+2 \ k+1)$.

**Theorem (i)** Any permutation is a product of transpositions. (ii) If $\pi = \tau_1\tau_2\ldots\tau_k$, where $\tau_i$ are transpositions, then the numbers $k$ and $N(\pi)$ are of the same parity.

*Proof: (i) Any permutation is a product of disjoint cycles. By Lemma 3, any cycle is a product of transpositions. (ii) By Lemma 3, each of $\tau_1, \tau_2, \ldots, \tau_k$ is a product of an odd number of adjacent transpositions. Hence $\pi = \tau'_1\tau'_2\ldots\tau'_m$, where $\tau'_i$ are adjacent transpositions and number $m$ is of the same parity as $k$. By Lemma 2, $m$ has the same parity as $N(\pi)$. 
Definition of determinant

**Definition.** \( \det (a) = a, \) \[
\begin{vmatrix}
a & b \\
c & d
\end{vmatrix}
= ad - bc,
\]

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
\]

If \( A = (a_{ij}) \) is an \( n \times n \) matrix then

\[
\det A = \sum_{\pi \in S(n)} \text{sgn}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},
\]

where \( \pi \) runs over all permutations of \( \{1, 2, \ldots, n\} \).
**Theorem** \( \det A^T = \det A \).

**Proof:** Let \( A = (a_{ij})_{1 \leq i, j \leq n} \). Then \( A^T = (b_{ij})_{1 \leq i, j \leq n} \), where \( b_{ij} = a_{ji} \). We have

\[
\det A^T = \sum_{\pi \in S(n)} \text{sgn}(\pi) b_{1,\pi(1)} b_{2,\pi(2)} \ldots b_{n,\pi(n)}
\]

\[
= \sum_{\pi \in S(n)} \text{sgn}(\pi) a_{\pi(1),1} a_{\pi(2),2} \ldots a_{\pi(n),n}
\]

\[
= \sum_{\pi \in S(n)} \text{sgn}(\pi) a_{1,\pi^{-1}(1)} a_{2,\pi^{-1}(2)} \ldots a_{n,\pi^{-1}(n)}.
\]

When \( \pi \) runs over all permutations of \( \{1, 2, \ldots, n\} \), so does \( \sigma = \pi^{-1} \). It follows that

\[
\det A^T = \sum_{\sigma \in S(n)} \text{sgn}(\sigma^{-1}) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)}
\]

\[
= \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{n,\sigma(n)} = \det A.
\]
**Theorem 1**  Suppose $A$ is a square matrix and $B$ is obtained from $A$ by exchanging two rows. Then $\det B = -\det A$.

**Theorem 2**  Suppose $A$ is a square matrix and $B$ is obtained from $A$ by permuting its rows. Then $\det B = \det A$ if the permutation is even and $\det B = -\det A$ if the permutation is odd.