MATH 304–503 Spring 2010

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Problem 2 (20 pts.) Let $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (1,0,1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(iii) Find the matrix of the operator L relative to the standard basis.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iv) Find a diagonal matrix X and an invertible matrix U such that $B = UXU^{-1}$.

Problem 5 (20 pts.) Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1,1,0,0)$, $\mathbf{x}_2 = (2,0,-1,1)$, and $\mathbf{x}_3 = (0,1,1,0)$.

(i) Find the distance from the point $\mathbf{y} = (0,0,0,4)$ to the subspace V.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

Bonus Problem 6 (15 pts.) (i) Find a matrix exponential $\exp(tC)$, where $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ and $t \in \mathbb{R}$.

(ii) Solve a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 3x + y, \\ \dfrac{dy}{dt} = 3y \end{cases}$$

subject to the initial conditions $x(0) = y(0) = 1$.

Bonus Problem 7 (15 pts.) Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = D\mathbf{x}, \quad \text{where} \quad D = \frac{1}{9}\begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

The operator K is a rotation about an axis.

(i) Find the axis of rotation.

(ii) Find the angle of rotation.