MATH 304

Linear Algebra

Lecture 2:

Gaussian elimination.
Row echelon form.
Gauss-Jordan reduction.

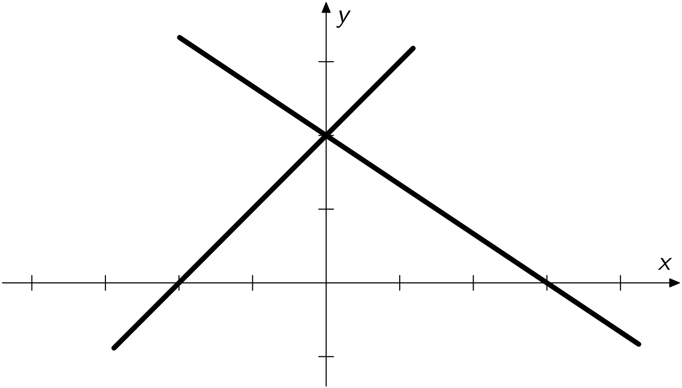
System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

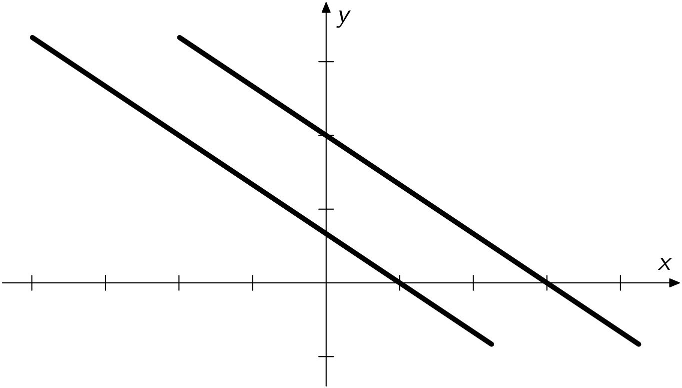
A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.



$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

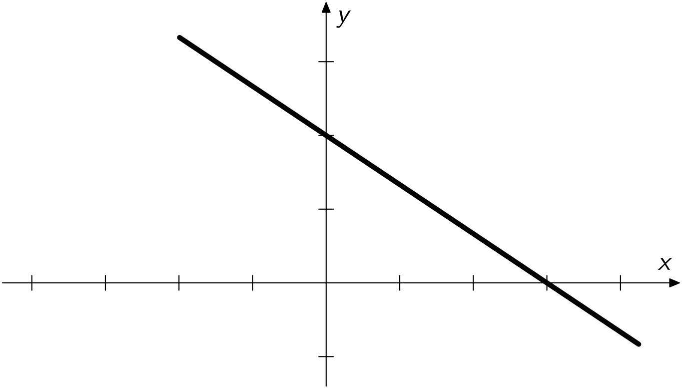
$$x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system

(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = 2 \implies x = y + 2$$

$$2x - y - z = 5 \implies 2(y+2) - y - z = 5$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 1 \implies x = y + 2 = 3$$

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Proposition Any elementary operation can be undone by applying another elementary operation.

Operation 1: multiply the ith equation by $r \neq 0$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

To undo the operation, multiply the ith equation by r^{-1}.

Operation 2: add r times the ith equation to the jth equation.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases} \implies$$

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ (a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, add $-r$ times the ith equation to the jth equation.

Operation 3: interchange the *i*th and *j*th equations.

$$\begin{cases} \qquad \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \qquad \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \qquad \cdots\cdots\cdots\cdots \end{cases}$$

$$\Longrightarrow \begin{cases} \qquad \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \qquad \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \qquad \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, apply it once more.

Proposition Any elementary operation can be undone by applying another elementary operation.

Theorem Applying elementary operations to a system of linear equations does not change the solution set of the system.

Proof: It is easy to see that after an elementary operation we do not lose any solution. Since the operation can be undone by another elementary operation, neither we get any garbage solutions.

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution. Both parts can be done by applying a finite number of **elementary operations**.

Example.

$$\left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{array}\right. \to \left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & 2y & + & z & = & 4 \end{array}\right.$$

$$\to \left\{\begin{array}{rcrcrcr} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & 3z & = & 6 \end{array}\right. \to \left\{\begin{array}{rcrcrcr} x & & & & & = & 3 \\ & & y & & & = & 1 \\ & & & & z & = & 2 \end{array}\right.$$

Another example.

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 1 \end{array}\right.$$

Add the 1st equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & 5y & - & 5z & = & 2 \end{array}\right.$$

Add -5 times the 2nd equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & -13 \end{array}\right.$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Yet another example.

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 14 \end{array}\right.$$

Add the 1st equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & 5y & - & 5z & = & 15 \end{array}\right.$$

Add -5 times the 2nd equation to the 3rd equation:

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right.$$

Add -1 times the 2nd equation to the 1st equation:

$$\left\{\begin{array}{rcrcrcr} x & & & - & z & = & -2 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 0 \end{array}\right. \iff \left\{\begin{array}{l} x = z - 2 \\ y = z + 3 \end{array}\right.$$

Here z is a *free variable*.

It follows that $\left\{\begin{array}{l} x = t - 2 \\ y = t + 3 \\ z = t \end{array}\right.$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \;\; t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $(\sqrt{2}, 0, -\sqrt{3}, 5)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions $=$ **(# of rows)** $\times$ **(# of columns)**

n-by-*n*: **square matrix**

n-by-1: **column vector**

1-by-*n*: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

Elementary operations for systems of linear equations correspond to *elementary row operations* for augmented matrices:

(1) to multiply a row by a nonzero scalar;

(2) to add the ith row multiplied by some $r \in \mathbb{R}$ to the jth row;

(3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Row echelon form

Definition. **Leading entry** of a matrix is the first nonzero entry in a row.

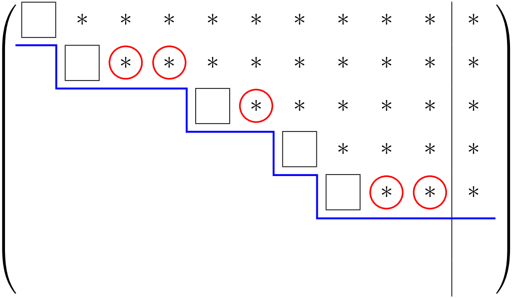
The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

$$\left(\begin{array}{cccccccccc|c} 1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\ 0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row echelon form

General matrix in row echelon form:

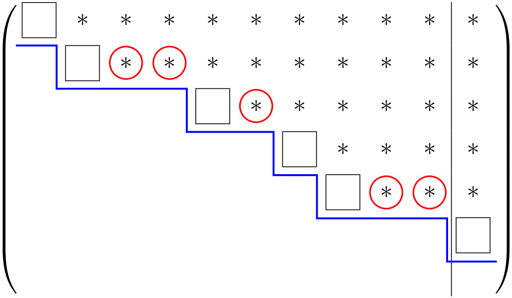


- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a free variable.

Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

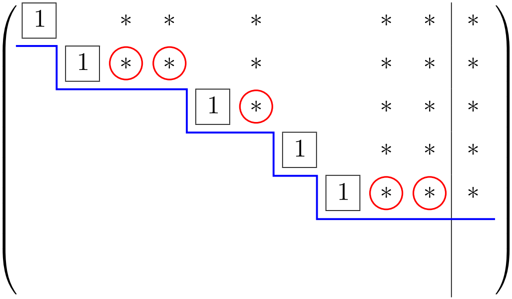
$$\left(\begin{array}{ccccccc|c}
\square & * & * & * & * & * & * & * \\
 & \square & * & * & * & * & * & * \\
 & & \square & * & * & * & * & * \\
 & & & \square & * & * & * & * \\
 & & & & \square & * & * & * \\
 & & & & & \square & * & * \\
 & & & & & & \square & *
\end{array}\right)$$

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.



Inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:



- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle marks a free variable.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \qquad \left(\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Another example.

$$\left\{\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ -x & + & 4y & - & 3z & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array}\right)$$

Row echelon form:

$$\left\{\begin{array}{rcrcrcl} x & + & y & - & 2z & = & 1 \\ & & y & - & z & = & 3 \\ & & & & 0 & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Reduced row echelon form:

$$\left\{\begin{array}{rcrcrcl} x & & & - & z & = & 0 \\ & & y & - & z & = & 0 \\ & & & & 0 & = & 1 \end{array}\right. \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \qquad \left(\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

New example. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$

Augmented matrix: $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$

The matrix is already in row echelon form.
To convert it into reduced row echelon form,
add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

x_3 and x_4 are free variables

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \Longleftrightarrow \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

System of linear equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$

$$= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$$

Example with a parameter.

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \qquad (a \in \mathbb{R}) \\ x + 2y + az = 0 \end{cases}$$

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent ($x = y = z = 0$ is a solution).

Augmented matrix: $\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:

$$\left(\begin{array}{ccr|c} 0 & 1 & 3 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right)$$

Now we can start the elimination.

First subtract the 1st row from the 3rd row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 2 & a & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right)$$

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & a+2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right)$$

At this point row reduction splits into two cases.

Case 1: $a \neq 1$. In this case, multiply the 3rd row by $(a-1)^{-1}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

The matrix is converted into row echelon form.
We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Add 2 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccr|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Finally, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Thus $x = y = z = 0$ is the only solution.

Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable.

$$\begin{cases} x - 5z = 0 \\ y + 3z = 0 \end{cases} \iff \begin{cases} x = 5z \\ y = -3z \end{cases}$$

System of linear equations:

$$\begin{cases} y + 3z = 0 \\ x + y - 2z = 0 \\ x + 2y + az = 0 \end{cases}$$

Solution: If $a \neq 1$ then $(x, y, z) = (0, 0, 0)$;
if $a = 1$ then $(x, y, z) = (5t, -3t, t)$, $t \in \mathbb{R}$.