

MATH 304
Linear Algebra

Lecture 10:
Linear independence.
Basis of a vector space.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \dots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in \mathbb{R}^3 .

$$\begin{aligned} x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} &\implies (x, y, z) = \mathbf{0} \\ &\implies x = y = z = 0 \end{aligned}$$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} aE_{11} + bE_{12} + cE_{21} + dE_{22} = O &\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O \\ &\implies a = b = c = d = 0 \end{aligned}$$

Examples of linear independence

- Polynomials $1, x, x^2, \dots, x^n$.

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0 \text{ identically}$$

$$\implies a_i = 0 \text{ for } 0 \leq i \leq n$$

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x - 1)^2$.

$$\begin{aligned} a_1p_1(x) + a_2p_2(x) + a_3p_3(x) &= a_1 + a_2(x - 1) + a_3(x - 1)^2 = \\ &= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3x^2. \end{aligned}$$

$$\text{Hence } a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0 \text{ identically}$$

$$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$$

$$\implies a_1 = a_2 = a_3 = 0$$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{v}_3 = (4, -7, 3)$. Determine whether vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular.

We obtain that $\det A = 0$.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k - 1$ vectors.

Proof: (i) \implies (ii) Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\frac{r_1}{r_i}\mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i}\mathbf{v}_k.$$

(ii) \implies (i) Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j . Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

Proof: Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \dots, a_{nj})$ for $j = 1, 2, \dots, m$. Then the vector equality $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \dots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \dots + a_{2m}t_m = 0, \\ \dots\dots\dots \\ a_{n1}t_1 + a_{n2}t_2 + \dots + a_{nm}t_m = 0. \end{cases}$$

Note that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are columns of the matrix (a_{ij}) . The number of leading entries in the row echelon form is at most n . If $m > n$ then there are free variables, therefore the zero solution is not unique.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in \mathbb{R}^3 .

Two vectors are linearly dependent if and only if they are parallel. Hence \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in \mathbb{R}^3 are always linearly dependent.

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Problem. Show that functions e^x , e^{2x} , and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$\begin{aligned}ae^x + be^{2x} + ce^{3x} &= 0, \\ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ae^x + 4be^{2x} + 9ce^{3x} &= 0.\end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\begin{aligned} \det A(x) &= e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix} \\ &= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix} \\ &= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0. \end{aligned}$$

Since the matrix $A(x)$ is invertible, we obtain

$$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$$

Wronskian

Let f_1, f_2, \dots, f_n be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \dots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \dots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \dots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions f_1, f_2, \dots, f_n are linearly independent in $C[a, b]$.

Theorem 1 Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_k x}$ are linearly independent.

Theorem 2 The set of functions

$$\{x^m e^{\lambda x} \mid \lambda \in \mathbb{R}, m = 0, 1, 2, \dots\}$$

is linearly independent.

Spanning set

Let S be a subset of a vector space V .

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S . If S is not empty then $W = \text{Span}(S)$ consists of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$ and $r_1, \dots, r_k \in \mathbb{R}$.

We say that the set S **spans** the subspace W or that S is a **spanning set** for W .

Remark. If S_1 is a spanning set for a vector space V and $S_1 \subset S_2 \subset V$, then S_2 is also a spanning set for V .

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V .

“Spanning set” means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$. “Linearly independent” implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r'_1\mathbf{v}_1 + r'_2\mathbf{v}_2 + \cdots + r'_k\mathbf{v}_k$$

$$\implies (r_1 - r'_1)\mathbf{v}_1 + (r_2 - r'_2)\mathbf{v}_2 + \cdots + (r_k - r'_k)\mathbf{v}_k = \mathbf{0}$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \cdots = r_k - r'_k = 0$$

Examples. • Standard basis for \mathbb{R}^n :

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots, \\ \mathbf{e}_n = (0, 0, 0, \dots, 0, 1).$$

Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n$.

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for \mathcal{P} , the space of all polynomials.

Bases for \mathbb{R}^n

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^n .

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span \mathbb{R}^n .

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:

- (i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for \mathbb{R}^n ;
- (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for \mathbb{R}^n ;
- (iii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in \mathbb{R}^3 .

Vectors \mathbf{v}_1 and \mathbf{v}_2 are linearly independent (as they are not parallel), but they do not span \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span \mathbb{R}^3 (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span \mathbb{R}^3), but they are linearly dependent.