MATH 304

Linear Algebra

Lecture 12:
Rank and nullity of a matrix.

Basis

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Equivalently, a subset $S \subset V$ is a basis for V if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices
$\dim \mathcal{M}_{2,2}(\mathbb{R}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m\times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree less than n
$\dim \mathcal{P}_n = n$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Row space of a matrix

Definition. The **row space** of an $m\times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A.

The dimension of the row space is called the **rank** of the matrix A.

Theorem 1 The rank of a matrix A is the maximal number of linearly independent rows in A.

Theorem 2 Elementary row operations do not change the row space of a matrix.

Theorem 3 If a matrix A is in row echelon form, then the nonzero rows of A are linearly independent.

Corollary The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Theorem Elementary row operations do not change the row space of a matrix.

Proof: Suppose that A and B are $m\times n$ matrices such that B is obtained from A by an elementary row operation. Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be the rows of A and $\mathbf{b}_1, \dots, \mathbf{b}_m$ be the rows of B. We have to show that $\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) = \mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m)$.

Observe that any row $\mathbf{b}_i$ of B belongs to $\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$. Indeed, either $\mathbf{b}_i = \mathbf{a}_j$ for some $1 \leq j \leq m$, or $\mathbf{b}_i = r\mathbf{a}_i$ for some scalar $r \neq 0$, or $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m) \subset \mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m)$.

Now the matrix A can also be obtained from B by an elementary row operation. By the above,

$$\mathrm{Span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \subset \mathrm{Span}(\mathbf{b}_1, \dots, \mathbf{b}_m).$$

Problem. Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert A to row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 1 & 0 \\ 0 & \color{red}{1} & 1 \\ 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for the row space of A. Thus the rank of A is 3.

It follows that the row space of A is the entire space $\mathbb{R}^3$.

Problem. Find a basis for the vector space V spanned by vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, $\mathbf{w}_3 = (2, 3, 1)$, and $\mathbf{w}_4 = (1, 1, 1)$.

The vector space V is the row space of a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for V.

Column space of a matrix

Definition. The **column space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by columns of A.

Theorem 1 The column space of a matrix A coincides with the row space of the transpose matrix A^T.

Theorem 2 Elementary column operations do not change the column space of a matrix.

Theorem 3 Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

Theorem 4 For any matrix, the row space and the column space have the same dimension.

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The column space of A coincides with the row space of A^T. To find a basis, we convert A^T to row echelon form:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 2 & 1 \\ 0 & \color{red}{1} & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} \end{pmatrix}$$

Vectors $(1, 0, 2, 1)$, $(0, 1, 1, 0)$, and $(0, 0, 0, 1)$ form a basis for the column space of A.

Problem. Find a basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Alternative solution: We already know from a previous problem that the rank of A is 3. It follows that the columns of A are linearly independent. Therefore these columns form a basis for the column space.

Nullspace of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A, denoted $N(A)$, is the set of all n-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}.}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with A as the coefficient matrix).

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Theorem The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

Proof: We have to show that $N(A)$ is nonempty, closed under addition, and closed under scaling.

First of all, $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in N(A) \implies N(A)$ is not empty.

Secondly, if $\mathbf{x}, \mathbf{y} \in N(A)$, i.e., if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}+\mathbf{y} \in N(A)$.

Thirdly, if $\mathbf{x} \in N(A)$, i.e., if $A\mathbf{x} = \mathbf{0}$, then for any $r \in \mathbb{R}$ one has $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0} \implies r\mathbf{x} \in N(A)$.

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A.

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Elementary row operations do not change the nullspace. Let us convert A to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

General element of $N(A)$:

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (t + 2s, -2t - 3s, t, s) \\ &= t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}. \end{aligned}$$

Vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$ form a basis for $N(A)$. Thus the nullity of the matrix A is 2.

rank + nullity

Theorem The rank of a matrix A plus the nullity of A equals the number of columns in A.

Sketch of the proof: The rank of A equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of A equals the number of free variables in the corresponding system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank+nullity is the number of all columns in the matrix A.

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Alternative solution: Clearly, the rows of A are linearly independent. Therefore the rank of A is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of A is 2.