MATH 304

Linear Algebra

Lecture 14:
Basis and coordinates.
Change of basis.
Linear transformations.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements (called the *dimension* of V).

Example. Vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$,..., $\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$ form a basis for $\mathbb{R}^n$ (called *standard*) since

$$(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Examples. • Coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$, ..., $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$ are $(x_1, x_2, \ldots, x_n)$.

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$ relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d).

• Coordinates of a polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathcal{P}_n$ relative to the basis $1, x, x^2, \ldots, x^{n-1}$ are $(a_0, a_1, \ldots, a_{n-1})$.

Vectors $\mathbf{u}_1=(2,1)$ and $\mathbf{u}_2=(3,1)$ form a basis for $\mathbb{R}^2$.

Problem 1. Find coordinates of the vector $\mathbf{v} = (7,4)$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{u}_1 + y\mathbf{u}_2 \iff \begin{cases} 2x + 3y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = 5 \\ y = -1 \end{cases}$$

Problem 2. Find the vector $\mathbf{w}$ whose coordinates with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$ are $(7,4)$.

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(2,1) + 4(3,1) = (26,11)$$

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y').

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.
In standard coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of coordinates in $\mathbb{R}^n$

The usual (standard) coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$,..., $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for $\mathbb{R}^n$ and $(x'_1, x'_2, \ldots, x'_n)$ be the coordinates of the same vector $\mathbf{v}$ with respect to this basis.

Problem 1. Given the standard coordinates $(x_1, x_2, \ldots, x_n)$, find the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$.

Problem 2. Given the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$, find the standard coordinates $(x_1, x_2, \ldots, x_n)$.

It turns out that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

The matrix $U = (u_{ij})$ does not depend on the vector $\mathbf{x}$.

Columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis.

U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

This solves Problem 2. To solve Problem 1, we have to use the inverse matrix U^{-1}, which is the transition matrix from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates (x', y', z') of $\mathbf{x}$ satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where U is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|rrr} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) = (I \mid U_0^{-1})$$

Thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Change of coordinates: general case

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{tikzcd} & V \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ \mathbb{R}^n \arrow[rr] & & \mathbb{R}^n \end{tikzcd}$$

The composition $g_2 \circ g_1^{-1}$ is a transformation of $\mathbb{R}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $p_1(x) = 1$, $p_2(x) = x + 1$, $p_3(x) = (x + 1)^2$ to the basis $q_1(x) = 1$, $q_2(x) = x$, $q_3(x) = x^2$ for the vector space $\mathcal{P}_3$.

We have to find coordinates of the polynomials p_1, p_2, p_3 with respect to the basis q_1, q_2, q_3:

$p_1(x) = 1 = q_1(x)$,
$p_2(x) = x + 1 = q_1(x) + q_2(x)$,
$p_3(x) = (x+1)^2 = x^2+2x+1 = q_1(x)+2q_2(x)+q_3(x)$.

Hence the transition matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

Thus the polynomial identity

$$a_1 + a_2(x+1) + a_3(x+1)^2 = b_1 + b_2 x + b_3 x^2$$

is equivalent to the relation

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates $\mathbf{x}$

Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1\mathbf{x}$

Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$U_2^{-1}U_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

A linear mapping $\ell : V \to \mathbb{R}$ is called a **linear functional** on V.

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \to V_2$ is called a **linear operator**.

Linear mapping = linear transformation = linear function

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Remark. A function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ if and only if $b = 0$.

Properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in

V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y} = L(\mathbf{x})+L(\mathbf{y})$,
$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x})$.

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})\cdot\mathbf{v}_0 = \mathbf{x}\cdot\mathbf{v}_0+\mathbf{y}\cdot\mathbf{v}_0 = \ell(\mathbf{x})+\ell(\mathbf{y})$,
$\ell(r\mathbf{x}) = (r\mathbf{x})\cdot\mathbf{v}_0 = r(\mathbf{x}\cdot\mathbf{v}_0) = r\ell(\mathbf{x})$.

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v}\times\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m\times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = f(a)$, where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R})$, $L(f) = gf$, where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R})$, $L(f) = f'$.

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$,
$D(rf) = (rf)' = rf' = rD(f)$.

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R}$, $\ell(f) = \int_a^b f(x)\,dx$, where $a, b \in \mathbb{R}$, $a < b$.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).