MATH 304

Linear Algebra

Lecture 15:
Kernel and range.
General linear equation.
Marix transformations.

Linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y} = L(\mathbf{x})+L(\mathbf{y})$,
$L(r\mathbf{x}) = s(r\mathbf{x}) = r(s\mathbf{x}) = rL(\mathbf{x})$.

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})\cdot\mathbf{v}_0 = \mathbf{x}\cdot\mathbf{v}_0+\mathbf{y}\cdot\mathbf{v}_0 = \ell(\mathbf{x})+\ell(\mathbf{y})$,
$\ell(r\mathbf{x}) = (r\mathbf{x})\cdot\mathbf{v}_0 = r(\mathbf{x}\cdot\mathbf{v}_0) = r\ell(\mathbf{x})$.

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v}\times\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m\times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : F(\mathbb{R}) \to \mathbb{R},\ \ \ell(f) = f(a),$ where $a \in \mathbb{R}$.

- *Multiplication by a fixed function*

$L : F(\mathbb{R}) \to F(\mathbb{R}),\ \ L(f) = gf,$ where $g \in F(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R}),\ \ L(f) = f'$.

$D(f + g) = (f + g)' = f' + g' = D(f) + D(g),$
$D(rf) = (rf)' = rf' = rD(f).$

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R},\ \ \ell(f) = \int_a^b f(x)\,dx,$ where $a, b \in \mathbb{R},\ \ a < b$.

Properties of linear mappings

- If a linear mapping $L : V \longrightarrow W$ is invertible then the inverse mapping $L^{-1} : W \longrightarrow V$ is also linear.

- If $L : V \longrightarrow W$ and $M : W \longrightarrow X$ are linear mappings then the composition $M \circ L : V \longrightarrow X$ is also linear.

- If $L_1 : V \longrightarrow W$ and $L_2 : V \longrightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Range and kernel

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L, denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range of L is a subspace of W.
(ii) The kernel of L is a subspace of V.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker L$ is the nullspace of the matrix.

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $f(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1,1,1)$, $(0,2,0)$, and $(-1,-1,-1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1,1,1)$ and $(0,1,0)$.

To find $\ker L$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x,y,z) \in \ker L$ if $x - z = y = 0$.
It follows that $\ker L$ is the line spanned by $(1,0,1)$.

More examples

$$f : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}), \;\; f(A) = A + A^T.$$

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}.$$

$\ker f$ is the subspace of anti-symmetric matrices, the range of f is the subspace of symmetric matrices.

$$g : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}), \;\; g(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A.$$

$$g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

The range of g is the subspace of matrices with the zero second row, $\ker g$ is the same as the range $\implies g(g(A)) = O$.

General linear equations

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for the kernel of L, and $t_1, \dots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$

$$L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 1 & 1 & 4 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} \color{red}{1} & 0 & 2 & 5 \\ 0 & \color{red}{1} & -1 & -1 \end{array}\right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u''(x) + u(x) = e^{2x}$.

Linear operator $L : C^2(\mathbb{R}) \to C(\mathbb{R})$, $Lu = u'' + u$.
Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

It can be shown that the range of L is the entire space $C(\mathbb{R})$ while the kernel of L is spanned by the functions $\sin x$ and $\cos x$.

Particular solution: $u_0 = \frac{1}{5}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{5}e^{2x} + t_1 \sin x + t_2 \cos x.$$

Matrix transformations

Any $m \times n$ matrix A gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $L(\mathbf{x}) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is **linear**.

Example. $$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(\mathbf{e}_1) = (1,3,0)$, $L(\mathbf{e}_2) = (0,4,5)$, $L(\mathbf{e}_3) = (2,7,8)$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$
$$= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3)$$
$$= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Change of coordinates

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & \overset{g_1}{\swarrow} & & \overset{g_2}{\searrow} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself. It is represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \to W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \big\downarrow & & \big\downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$.