

MATH 304
Linear Algebra

Lecture 20:
The Gram-Schmidt process (continued).
Eigenvalues and eigenvectors.

Orthogonal sets

Let V be a vector space with an inner product.

Definition. Nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ form an **orthogonal set** if they are orthogonal to each other: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$.

If, in addition, all vectors are of unit norm, $\|\mathbf{v}_i\| = 1$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is called an **orthonormal set**.

Theorem Any orthogonal set is linearly independent.

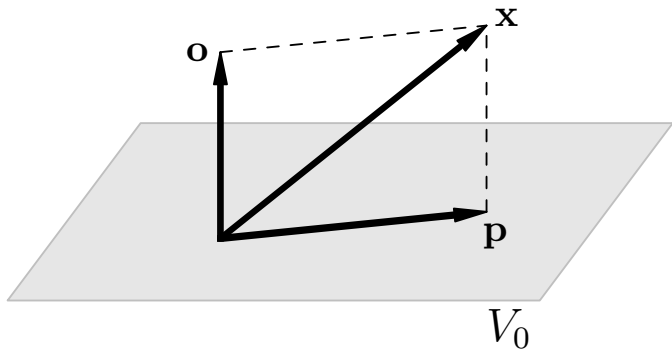
Orthogonal projection

Theorem Let V be a finite-dimensional inner product space and V_0 be a subspace of V . Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

The component \mathbf{p} is the **orthogonal projection** of the vector \mathbf{x} onto the subspace V_0 . The distance from \mathbf{x} to the subspace V_0 is $\|\mathbf{o}\|$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V_0 then

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n.$$



The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for V . Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

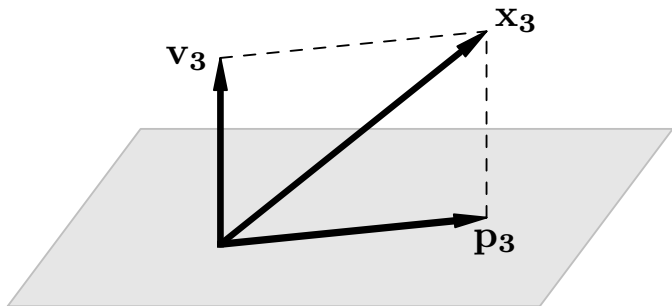
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

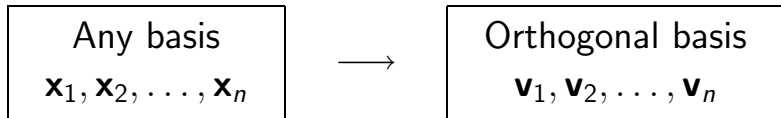
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$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \dots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V .



$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$$



Properties of the Gram-Schmidt process:

- $\mathbf{v}_k = \mathbf{x}_k - (\alpha_1 \mathbf{x}_1 + \dots + \alpha_{k-1} \mathbf{x}_{k-1})$, $1 \leq k \leq n$;
- the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the same as the span of $\mathbf{x}_1, \dots, \mathbf{x}_k$;
- \mathbf{v}_k is orthogonal to $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$, where \mathbf{p}_k is the orthogonal projection of the vector \mathbf{x}_k on the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$;
- $\|\mathbf{v}_k\|$ is the distance from \mathbf{x}_k to the subspace spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$.

Normalization

Let V be a vector space with an inner product.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V .

Let $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, \dots , $\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.

Then $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthonormal basis for V .

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Orthogonalization / Normalization

An alternative form of the Gram-Schmidt process combines orthogonalization with normalization.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for an inner product space V . Let

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{w}_1 \rangle \mathbf{w}_1, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{x}_3, \mathbf{w}_2 \rangle \mathbf{w}_2, \quad \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|},$$

.....

$$\mathbf{v}_n = \mathbf{x}_n - \langle \mathbf{x}_n, \mathbf{w}_1 \rangle \mathbf{w}_1 - \dots - \langle \mathbf{x}_n, \mathbf{w}_{n-1} \rangle \mathbf{w}_{n-1},$$

$$\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}.$$

Then $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthonormal basis for V .

Problem. Let Π be the plane in \mathbb{R}^3 spanned by vectors $\mathbf{x}_1 = (1, 2, 2)$ and $\mathbf{x}_2 = (-1, 0, 2)$.

- (i) Find an orthonormal basis for Π .
- (ii) Extend it to an orthonormal basis for \mathbb{R}^3 .

$\mathbf{x}_1, \mathbf{x}_2$ is a basis for the plane Π . We can extend it to a basis for \mathbb{R}^3 by adding one vector from the standard basis. For instance, vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3 = (0, 0, 1)$ form a basis for \mathbb{R}^3 because

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0.$$

Using the Gram-Schmidt process, we orthogonalize the basis $\mathbf{x}_1 = (1, 2, 2)$, $\mathbf{x}_2 = (-1, 0, 2)$, $\mathbf{x}_3 = (0, 0, 1)$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 2, 2),$$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (-1, 0, 2) - \frac{3}{9}(1, 2, 2) \\ &= (-4/3, -2/3, 4/3),\end{aligned}$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\ &= (0, 0, 1) - \frac{2}{9}(1, 2, 2) - \frac{4/3}{4}(-4/3, -2/3, 4/3) \\ &= (2/9, -2/9, 1/9).\end{aligned}$$

Now $\mathbf{v}_1 = (1, 2, 2)$, $\mathbf{v}_2 = (-4/3, -2/3, 4/3)$,
 $\mathbf{v}_3 = (2/9, -2/9, 1/9)$ is an orthogonal basis for \mathbb{R}^3
while $\mathbf{v}_1, \mathbf{v}_2$ is an orthogonal basis for Π . It remains
to normalize these vectors.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 9 \implies \|\mathbf{v}_1\| = 3$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 4 \implies \|\mathbf{v}_2\| = 2$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = 1/9 \implies \|\mathbf{v}_3\| = 1/3$$

$$\mathbf{w}_1 = \mathbf{v}_1 / \|\mathbf{v}_1\| = (1/3, 2/3, 2/3) = \frac{1}{3}(1, 2, 2),$$

$$\mathbf{w}_2 = \mathbf{v}_2 / \|\mathbf{v}_2\| = (-2/3, -1/3, 2/3) = \frac{1}{3}(-2, -1, 2),$$

$$\mathbf{w}_3 = \mathbf{v}_3 / \|\mathbf{v}_3\| = (2/3, -2/3, 1/3) = \frac{1}{3}(2, -2, 1).$$

$\mathbf{w}_1, \mathbf{w}_2$ is an orthonormal basis for Π .

$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is an orthonormal basis for \mathbb{R}^3 .

Problem. Find the distance from the point $\mathbf{y} = (0, 0, 0, 1)$ to the subspace $\Pi \subset \mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, -1, 1, -1)$, $\mathbf{x}_2 = (1, 1, 3, -1)$, and $\mathbf{x}_3 = (-3, 7, 1, 3)$.

Let us apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$. We should obtain an orthogonal system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. The desired distance will be $|\mathbf{v}_4|$.

$$\mathbf{x}_1 = (1, -1, 1, -1), \quad \mathbf{x}_2 = (1, 1, 3, -1), \\ \mathbf{x}_3 = (-3, 7, 1, 3), \quad \mathbf{y} = (0, 0, 0, 1).$$

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1) \\ = (0, 2, 2, 0),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\ = (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0) \\ = (0, 0, 0, 0).$$

The Gram-Schmidt process can be used to check linear independence of vectors!

The vector \mathbf{x}_3 is a linear combination of \mathbf{x}_1 and \mathbf{x}_2 .

Π is a plane, not a 3-dimensional subspace.

We should orthogonalize vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$.

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\ &= (0, 0, 0, 1) - \frac{-1}{4}(1, -1, 1, -1) - \frac{0}{8}(0, 2, 2, 0) \\ &= (1/4, -1/4, 1/4, 3/4).\end{aligned}$$

$$|\mathbf{v}_4| = \left| \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right) \right| = \frac{1}{4} |(1, -1, 1, 3)| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.$$

Problem. Find the distance from the point $\mathbf{z} = (0, 0, 1, 0)$ to the plane Π that passes through the point $\mathbf{x}_0 = (1, 0, 0, 0)$ and is parallel to the vectors $\mathbf{v}_1 = (1, -1, 1, -1)$ and $\mathbf{v}_2 = (0, 2, 2, 0)$.

The plane Π is not a subspace of \mathbb{R}^4 as it does not pass through the origin. Let $\Pi_0 = \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then $\Pi = \Pi_0 + \mathbf{x}_0$.

Hence the distance from the point \mathbf{z} to the plane Π is the same as the distance from the point $\mathbf{z} - \mathbf{x}_0$ to the plane Π_0 .

We shall apply the Gram-Schmidt process to vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{z} - \mathbf{x}_0$. This will yield an orthogonal system $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. The desired distance will be $|\mathbf{w}_3|$.

$$\mathbf{v}_1 = (1, -1, 1, -1), \mathbf{v}_2 = (0, 2, 2, 0), \mathbf{z} - \mathbf{x}_0 = (-1, 0, 1, 0).$$

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, -1, 1, -1),$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 = \mathbf{v}_2 = (0, 2, 2, 0) \text{ as } \mathbf{v}_2 \perp \mathbf{v}_1.$$

$$\mathbf{w}_3 = (\mathbf{z} - \mathbf{x}_0) - \frac{\langle \mathbf{z} - \mathbf{x}_0, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{z} - \mathbf{x}_0, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2$$

$$= (-1, 0, 1, 0) - \frac{0}{4}(1, -1, 1, -1) - \frac{2}{8}(0, 2, 2, 0)$$

$$= (-1, -1/2, 1/2, 0).$$

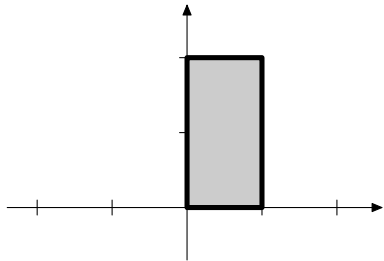
$$|\mathbf{w}_3| = \left| \left(-1, -\frac{1}{2}, \frac{1}{2}, 0 \right) \right| = \frac{1}{2} |(-2, -1, 1, 0)| = \frac{\sqrt{6}}{2} = \sqrt{\frac{3}{2}}.$$

Linear transformations of \mathbb{R}^2

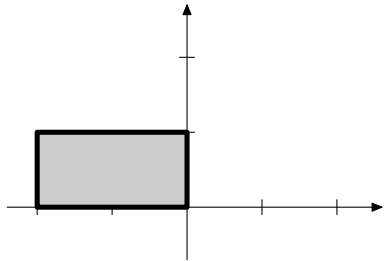
Any linear mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

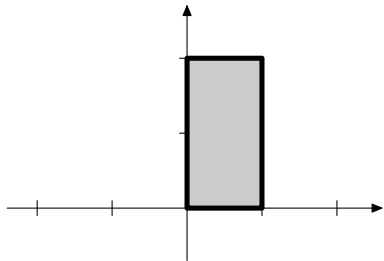
Linear transformations corresponding to particular matrices can have various geometric properties.



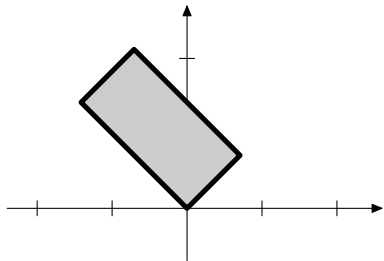
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



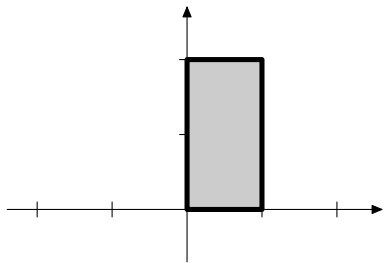
Rotation by 90°



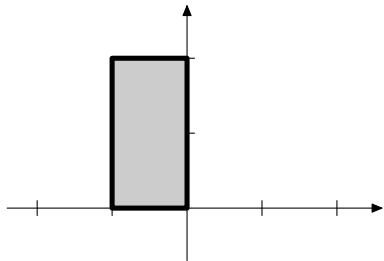
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



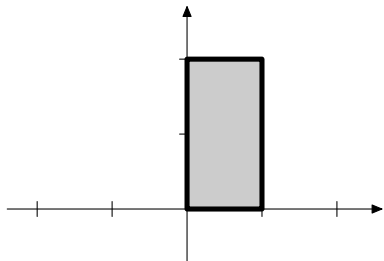
Rotation by 45°



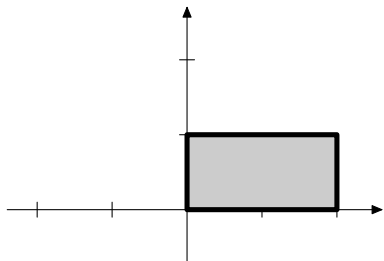
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



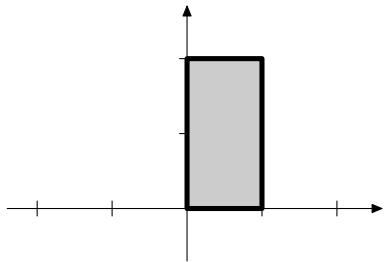
Reflection in
the vertical axis



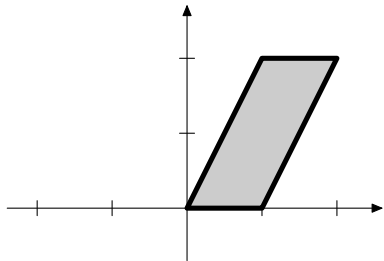
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



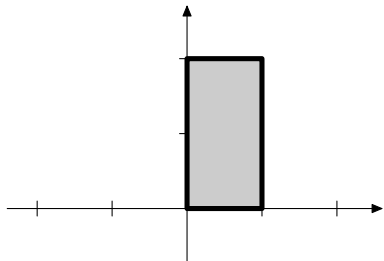
Reflection in
the line $x - y = 0$



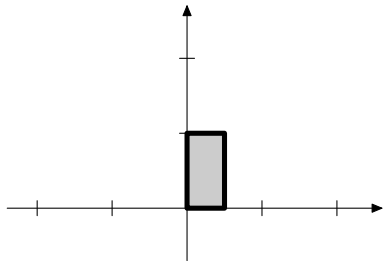
$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$



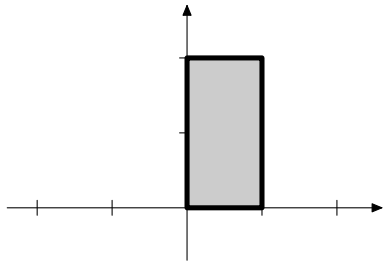
Horizontal shear



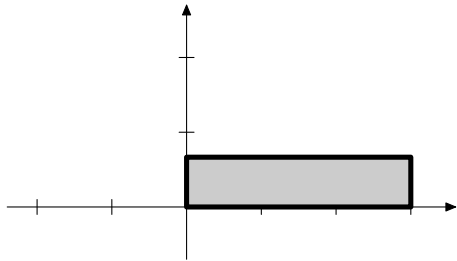
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



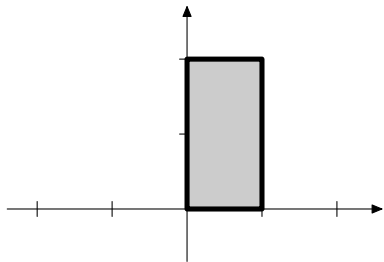
Scaling



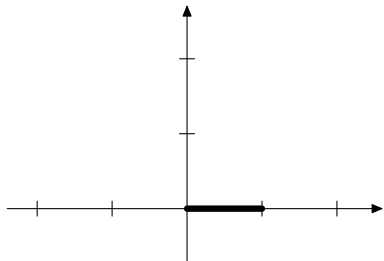
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



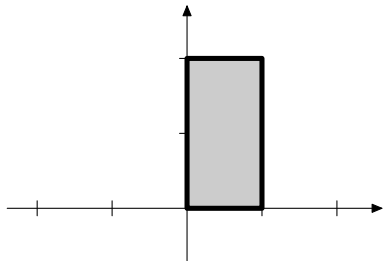
Squeeze



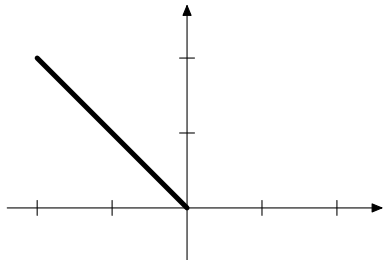
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



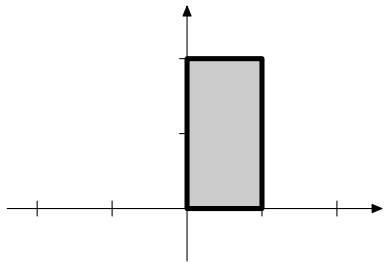
Vertical projection on
the horizontal axis



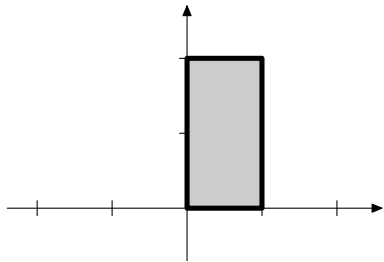
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity

Eigenvalues and eigenvectors of a matrix

Definition. Let A be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix A if

$A\mathbf{v} = \lambda\mathbf{v}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$.

The vector \mathbf{v} is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ .

Remarks.

- Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

- The zero vector is never considered an eigenvector.

Example. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence $(1, 0)$ is an eigenvector of A belonging to the eigenvalue 2, while $(0, -2)$ is an eigenvector of A belonging to the eigenvalue 3.