MATH 304

Linear Algebra

**Lecture 2:
Gaussian elimination.**

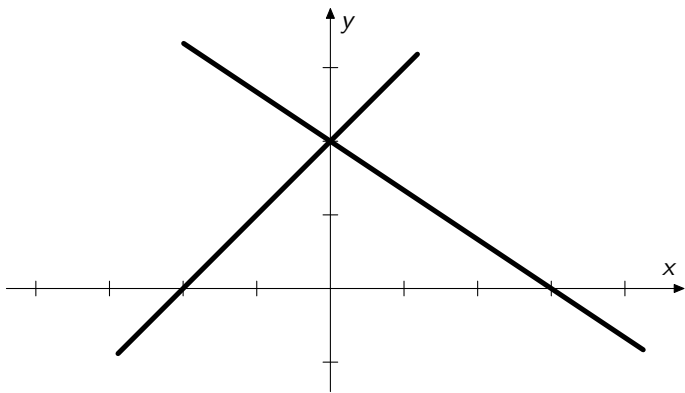## System of linear equations

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\qquad \cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
$$

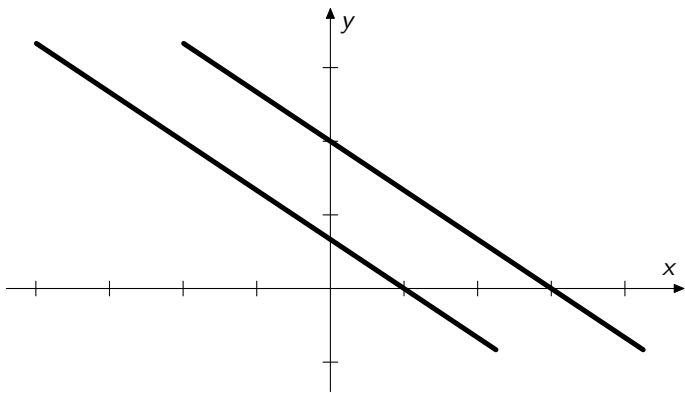Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

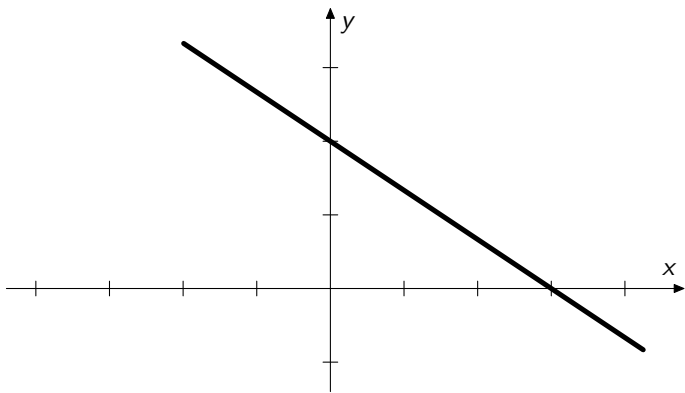$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \qquad x = 0, \ y = 2$$

$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$ *inconsistent system* (no solutions)

$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = 2 \implies x = y + 2$$

$$2x - y - z = 5 \implies 2(y + 2) - y - z = 5$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 1 \implies x = y + 2 = 3$$

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

After elimination:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

After back substitution:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

After elimination:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 0 = 0 \end{cases}$$

After back substitution:

$$\begin{cases} x = t - 2 \\ y = t + 3 \qquad (t \in \mathbb{R}) \\ z = t \end{cases}$$

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Theorem (i) Applying elementary operations to a system of linear equations does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation.

Operation 1: multiply the *i*th equation by $r \neq 0$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \qquad \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \qquad \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\implies \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \qquad \cdots\cdots\cdots\cdots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \qquad \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

To undo the operation, multiply the *i*th equation by r^{-1}.

Operation 2: add r times the ith equation to the jth equation.

$$
\begin{cases}
\quad\quad \ldots\ldots\ldots\ldots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\quad\quad \ldots\ldots\ldots\ldots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
\quad\quad \ldots\ldots\ldots\ldots
\end{cases}
\implies
$$

$$
\begin{cases}
\quad\quad \ldots\ldots\ldots\ldots \\
a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\
\quad\quad \ldots\ldots\ldots\ldots \\
(a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\
\quad\quad \ldots\ldots\ldots\ldots
\end{cases}
$$

To undo the operation, add $-r$ times the ith equation to the jth equation.

Operation 3: interchange the ith and jth equations.

$$\begin{cases} \quad \cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \quad \cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \quad \cdots\cdots\cdots \end{cases}$$

$$\Longrightarrow \begin{cases} \quad \cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \quad \cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \quad \cdots\cdots\cdots \end{cases}$$

To undo the operation, apply it once more.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Add -2 times the 1st equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y - z = -1 \\ x + y + z = 6 \end{cases} \qquad \boxed{R2 := R2 - 2 * R1}$$

Add -1 times the 1st equation to the 3rd equation:

$$\begin{cases} x - y & = 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Add -2 times the 2nd equation to the 3rd equation:

$$\begin{cases} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & 3z & = & 6 \end{cases}$$

The elimination is completed, and we can solve the system by back substitution. However we can as well proceed with elementary operations.

Multiply the 3rd equation by $1/3$:

$$\begin{cases} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & z & = & 2 \end{cases}$$

Add the 3rd equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y & = 1 \\ z & = 2 \end{cases}$$

Add the 2nd equation to the 1st equation:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases}$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x & - & z & = & -2 \\ & y & - & z & = & 3 \\ & & 0 & = & 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that $\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases}$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \ \ t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

The set of all solutions is a straight line in $\mathbb{R}^3$ passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $(\sqrt{2}, 0, -\sqrt{3}, 5)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions = (# of rows) × (# of columns)

$n \times n$: **square matrix**

$n \times 1$: **column vector**

$1 \times n$: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array} \right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the augmented matrix.

Elementary operations for systems of linear equations correspond to *elementary row operations* for augmented matrices:

(1) to multiply a row by a nonzero scalar;

(2) to add the ith row multiplied by some $r \in \mathbb{R}$ to the jth row;

(3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right) = \left(\begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{array}\right),$$

where $\mathbf{v}_i = (a_{i1} \; a_{i2} \; \ldots \; a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$
\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}
$$

Elementary row operations

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$
\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}
$$