

MATH 304  
Linear Algebra

**Lecture 11:**  
**Vector spaces.**

## Linear operations on vectors

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  be  $n$ -dimensional vectors, and  $r \in \mathbb{R}$  be a scalar.

*Vector sum:*  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

*Scalar multiple:*  $r\mathbf{x} = (rx_1, rx_2, \dots, rx_n)$

*Zero vector:*  $\mathbf{0} = (0, 0, \dots, 0)$

*Negative of a vector:*  $-\mathbf{y} = (-y_1, -y_2, \dots, -y_n)$

*Vector difference:*

$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$

## Properties of linear operations

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$$

$$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$$

$$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$$

$$(rs)\mathbf{x} = r(s\mathbf{x})$$

$$1\mathbf{x} = \mathbf{x}$$

$$0\mathbf{x} = \mathbf{0}$$

$$(-1)\mathbf{x} = -\mathbf{x}$$

## Linear operations on matrices

Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be  $m \times n$  matrices, and  $r \in \mathbb{R}$  be a scalar.

*Matrix sum:*  $A + B = (a_{ij} + b_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$

*Scalar multiple:*  $rA = (ra_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$

*Zero matrix  $O$ :* all entries are zeros

*Negative of a matrix:*  $-A = (-a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$

*Matrix difference:*  $A - B = (a_{ij} - b_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$

As far as the linear operations are concerned, the  $m \times n$  matrices have the same properties as  $mn$ -dimensional vectors.

## Vector space: informal description

*Vector space = linear space* = a set  $V$  of objects (called *vectors*) that can be added and scaled.

That is, for any  $\mathbf{u}, \mathbf{v} \in V$  and  $r \in \mathbb{R}$  expressions

$$\boxed{\mathbf{u} + \mathbf{v}} \text{ and } \boxed{r\mathbf{u}}$$

should make sense.

Certain restrictions apply. For instance,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$

$$2\mathbf{u} + 3\mathbf{u} = 5\mathbf{u}.$$

That is, addition and scalar multiplication in  $V$  should be like those of  $n$ -dimensional vectors.

## Vector space: definition

*Vector space* is a set  $V$  equipped with two operations  $\alpha : V \times V \rightarrow V$  and  $\mu : \mathbb{R} \times V \rightarrow V$  that have certain properties (listed below).

The operation  $\alpha$  is called *addition*. For any  $\mathbf{u}, \mathbf{v} \in V$ , the element  $\alpha(\mathbf{u}, \mathbf{v})$  is denoted  $\mathbf{u} + \mathbf{v}$ .

The operation  $\mu$  is called *scalar multiplication*. For any  $r \in \mathbb{R}$  and  $\mathbf{u} \in V$ , the element  $\mu(r, \mathbf{u})$  is denoted  $r\mathbf{u}$ .

## Properties of addition and scalar multiplication (brief)

A1.  $\mathbf{a + b = b + a}$

A2.  $\mathbf{(a + b) + c = a + (b + c)}$

A3.  $\mathbf{a + 0 = 0 + a = a}$

A4.  $\mathbf{a + (-a) = (-a) + a = 0}$

A5.  $\mathbf{r(a + b) = ra + rb}$

A6.  $\mathbf{(r + s)a = ra + sa}$

A7.  $\mathbf{(rs)a = r(sa)}$

A8.  $\mathbf{1a = a}$

## Properties of addition and scalar multiplication (detailed)

**A1.**  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$  for all  $\mathbf{a}, \mathbf{b} \in V$ .

**A2.**  $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$  for all  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ .

**A3.** There exists an element of  $V$ , called the *zero vector* and denoted  $\mathbf{0}$ , such that  $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$  for all  $\mathbf{a} \in V$ .

**A4.** For any  $\mathbf{a} \in V$  there exists an element of  $V$ , denoted  $-\mathbf{a}$ , such that  $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ .

**A5.**  $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$  for all  $r \in \mathbb{R}$  and  $\mathbf{a}, \mathbf{b} \in V$ .

**A6.**  $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$  for all  $r, s \in \mathbb{R}$  and  $\mathbf{a} \in V$ .

**A7.**  $(rs)\mathbf{a} = r(s\mathbf{a})$  for all  $r, s \in \mathbb{R}$  and  $\mathbf{a} \in V$ .

**A8.**  $1\mathbf{a} = \mathbf{a}$  for all  $\mathbf{a} \in V$ .



- Associativity of addition implies that a multiple sum  $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$  is well defined for any  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in V$ .

- **Subtraction** in  $V$  is defined as usual:  
 $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ .

- Addition and scalar multiplication are called **linear operations**.

Given  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in V$  and  $r_1, r_2, \dots, r_k \in \mathbb{R}$ ,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ .

## Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties A1–A8 are easy to verify.

- $\mathbb{R}^n$ :  $n$ -dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$ :  $m \times n$  matrices with real entries

- $\mathbb{R}^\infty$ : infinite sequences  $(x_1, x_2, \dots)$ ,  $x_i \in \mathbb{R}$

For any  $\mathbf{x} = (x_1, x_2, \dots)$ ,  $\mathbf{y} = (y_1, y_2, \dots) \in \mathbb{R}^\infty$  and  $r \in \mathbb{R}$  let  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots)$ ,  $r\mathbf{x} = (rx_1, rx_2, \dots)$ . Then  $\mathbf{0} = (0, 0, \dots)$  and  $-\mathbf{x} = (-x_1, -x_2, \dots)$ .

- $\{\mathbf{0}\}$ : the trivial vector space

$$\mathbf{0} + \mathbf{0} = \mathbf{0}, \quad r\mathbf{0} = \mathbf{0}, \quad -\mathbf{0} = \mathbf{0}.$$

## Functional vector spaces

- $F(\mathbb{R})$ : the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

Given functions  $f, g \in F(\mathbb{R})$  and a scalar  $r \in \mathbb{R}$ , let  $(f + g)(x) = f(x) + g(x)$  and  $(rf)(x) = rf(x)$  for all  $x \in \mathbb{R}$ .  
Zero vector:  $o(x) = 0$ . Negative:  $(-f)(x) = -f(x)$ .

- $C(\mathbb{R})$ : all continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

Linear operations are inherited from  $F(\mathbb{R})$ . We only need to check that  $f, g \in C(\mathbb{R}) \implies f+g, rf \in C(\mathbb{R})$ , the zero function is continuous, and  $f \in C(\mathbb{R}) \implies -f \in C(\mathbb{R})$ .

- $C^1(\mathbb{R})$ : all continuously differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

- $C^\infty(\mathbb{R})$ : all smooth functions  $f : \mathbb{R} \rightarrow \mathbb{R}$

- $\mathcal{P}$ : all polynomials  $p(x) = a_0 + a_1x + \cdots + a_nx^n$

## Some general observations

- The zero vector is unique.

If  $\mathbf{z}_1$  and  $\mathbf{z}_2$  are zero vectors then  $\mathbf{z}_1 = \mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_2$ .

- For any  $\mathbf{a} \in V$ , the negative  $-\mathbf{a}$  is unique.

Suppose  $\mathbf{b}$  and  $\mathbf{b}'$  are negatives of  $\mathbf{a}$ . Then

$$\mathbf{b}' = \mathbf{b}' + \mathbf{0} = \mathbf{b}' + (\mathbf{a} + \mathbf{b}) = (\mathbf{b}' + \mathbf{a}) + \mathbf{b} = \mathbf{0} + \mathbf{b} = \mathbf{b}.$$

- $0\mathbf{a} = \mathbf{0}$  for any  $\mathbf{a} \in V$ .

Indeed,  $0\mathbf{a} + \mathbf{a} = 0\mathbf{a} + 1\mathbf{a} = (0 + 1)\mathbf{a} = 1\mathbf{a} = \mathbf{a}$ .

Then  $0\mathbf{a} + \mathbf{a} = \mathbf{a} \implies 0\mathbf{a} + \mathbf{a} - \mathbf{a} = \mathbf{a} - \mathbf{a} \implies 0\mathbf{a} = \mathbf{0}$ .

- $(-1)\mathbf{a} = -\mathbf{a}$  for any  $\mathbf{a} \in V$ .

Indeed,  $\mathbf{a} + (-1)\mathbf{a} = (-1)\mathbf{a} + \mathbf{a} = (-1)\mathbf{a} + 1\mathbf{a} = (-1 + 1)\mathbf{a} = 0\mathbf{a} = \mathbf{0}$ .

## Counterexample: dumb scaling

Consider the set  $V = \mathbb{R}^n$  with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{a} = \mathbf{0}} \quad \text{for any } \mathbf{a} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

$$\text{A5. } r \odot (\mathbf{a} + \mathbf{b}) = r \odot \mathbf{a} + r \odot \mathbf{b} \quad \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$$

$$\text{A6. } (r + s) \odot \mathbf{a} = r \odot \mathbf{a} + s \odot \mathbf{a} \quad \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$$

$$\text{A7. } (rs) \odot \mathbf{a} = r \odot (s \odot \mathbf{a}) \quad \iff \mathbf{0} = \mathbf{0}$$

$$\text{A8. } 1 \odot \mathbf{a} = \mathbf{a} \quad \iff \mathbf{0} = \mathbf{a}$$

A8 is the only property that fails. As a consequence, property A8 does not follow from properties A1–A7.

## Counterexample: lazy scaling

Consider the set  $V = \mathbb{R}^n$  with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{a} = \mathbf{a}} \text{ for any } \mathbf{a} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

$$\text{A5. } r \odot (\mathbf{a} + \mathbf{b}) = r \odot \mathbf{a} + r \odot \mathbf{b} \iff \mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{b}$$

$$\text{A6. } (r + s) \odot \mathbf{a} = r \odot \mathbf{a} + s \odot \mathbf{a} \iff \mathbf{a} = \mathbf{a} + \mathbf{a}$$

$$\text{A7. } (rs) \odot \mathbf{a} = r \odot (s \odot \mathbf{a}) \iff \mathbf{a} = \mathbf{a}$$

$$\text{A8. } 1 \odot \mathbf{a} = \mathbf{a} \iff \mathbf{a} = \mathbf{a}$$

The only property that fails is A6.