MATH 304

Linear Algebra

Lecture 18:
Rank and nullity of a matrix.
Basis and coordinates.
Change of coordinates.

Rank of a matrix

Definition. The **row space** of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A. The **column space** of A is a subspace of $\mathbb{R}^m$ spanned by columns of A.

The row space and the column space of A have the same dimension, which is called the **rank** of A.

Theorem 1 Elementary row operations do not change the row space of a matrix.

Theorem 2 If a matrix A is in row echelon form, then the nonzero rows of A form a basis for the row space.

Theorem 3 The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Nullspace of a matrix

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Definition. The **nullspace** of the matrix A, denoted $N(A)$, is the set of all n-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}.}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with A as the coefficient matrix).

Let A be an $m \times n$ matrix. Then the nullspace $N(A)$ is the solution set of a system of linear homogeneous equations in n variables.

Theorem The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

Definition. The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix A.

Problem. Find the nullity of the matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Elementary row operations do not change the nullspace.
Let us convert A to reduced row echelon form:
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$
$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

General element of $N(A)$:
$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (t + 2s, -2t - 3s, t, s) \\ &= t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}. \end{aligned}$$

Vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$ form a basis for $N(A)$.
Thus the nullity of the matrix A is 2.

rank + nullity

Theorem The rank of a matrix A plus the nullity of A equals the number of columns in A.

Sketch of the proof: The rank of A equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of A equals the number of free variables in the corresponding system, which equals the number of columns without leading entries in the row echelon form.

Consequently, rank+nullity is the number of all columns in the matrix A.

Problem. Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Alternative solution: Clearly, the rows of A are linearly independent. Therefore the rank of A is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of A is 2.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements (called the *dimension* of V).

Example. Vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$,..., $\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$ form a basis for $\mathbb{R}^n$ (called *standard*) since

$$(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between V and $\mathbb{R}^n$. This correspondence respects linear operations in V and in $\mathbb{R}^n$.

Examples. • Coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$,…, $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$ are $(x_1, x_2, \ldots, x_n)$.

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$ relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d).

• Coordinates of a polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathcal{P}_n$ relative to the basis $1, x, x^2, \ldots, x^{n-1}$ are $(a_0, a_1, \ldots, a_{n-1})$.

Vectors $\mathbf{u}_1{=}(3,1)$ and $\mathbf{u}_2{=}(2,1)$ form a basis for $\mathbb{R}^2$.

Problem 1. Find coordinates of the vector $\mathbf{v} = (7,4)$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$.

The desired coordinates x, y satisfy

$$\mathbf{v} = x\mathbf{u}_1 {+} y\mathbf{u}_2 \iff \begin{cases} 3x + 2y = 7 \\ x + y = 4 \end{cases} \iff \begin{cases} x = -1 \\ y = 5 \end{cases}$$

Problem 2. Find the vector $\mathbf{w}$ whose coordinates with respect to the basis $\mathbf{u}_1, \mathbf{u}_2$ are $(7,4)$.

$$\mathbf{w} = 7\mathbf{u}_1 + 4\mathbf{u}_2 = 7(3,1) + 4(2,1) = (29,11)$$

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{u}_1 = (3, 1)$, $\mathbf{u}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y').

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{u}_1 + y'\mathbf{u}_2$.
In standard coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of coordinates in $\mathbb{R}^n$

The usual (standard) coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$,..., $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for $\mathbb{R}^n$ and $(x'_1, x'_2, \ldots, x'_n)$ be the coordinates of the same vector $\mathbf{v}$ with respect to this basis.

Problem 1. Given the standard coordinates $(x_1, x_2, \ldots, x_n)$, find the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$.

Problem 2. Given the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$, find the standard coordinates $(x_1, x_2, \ldots, x_n)$.

It turns out that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}.$$

The matrix $U = (u_{ij})$ does not depend on the vector $\mathbf{v}$.

Columns of U are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis.

U is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

This solves Problem 2. To solve Problem 1, we have to use the inverse matrix U^{-1}, which is the transition matrix from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Problem. Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates (x', y', z') of $\mathbf{x}$ satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where U is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) = (I \mid U_0^{-1})$$

Thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$