

MATH 304

Linear Algebra

**Lecture 28:**

**Orthogonal bases.**

**The Gram-Schmidt orthogonalization process.**

## Orthogonal sets

Let  $V$  be an inner product space with an inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ .

*Definition.* Nonzero vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$  form an **orthogonal set** if they are orthogonal to each other:  $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$  for  $i \neq j$ .

If, in addition, all vectors are of unit norm,  $\|\mathbf{v}_i\| = 1$ , then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  is called an **orthonormal set**.

**Theorem** Any orthogonal set is linearly independent.

## Orthonormal bases

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be an orthonormal basis for an inner product space  $V$ .

**Theorem** Let  $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n$  and  $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \dots + y_n\mathbf{v}_n$ , where  $x_i, y_j \in \mathbb{R}$ . Then

(i)  $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$ ,

(ii)  $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ .

*Proof:* (ii) follows from (i) when  $\mathbf{y} = \mathbf{x}$ .

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle &= \left\langle \sum_{i=1}^n x_i \mathbf{v}_i, \sum_{j=1}^n y_j \mathbf{v}_j \right\rangle = \sum_{i=1}^n x_i \left\langle \mathbf{v}_i, \sum_{j=1}^n y_j \mathbf{v}_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^n x_i y_i.\end{aligned}$$

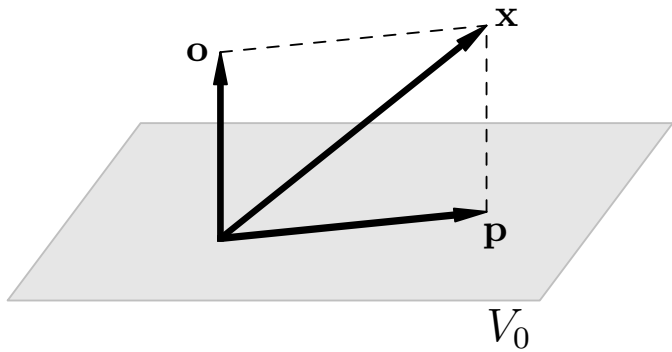
## Orthogonal projection

**Theorem** Let  $V$  be an inner product space and  $V_0$  be a finite-dimensional subspace of  $V$ . Then any vector  $\mathbf{x} \in V$  is uniquely represented as  $\mathbf{x} = \mathbf{p} + \mathbf{o}$ , where  $\mathbf{p} \in V_0$  and  $\mathbf{o} \perp V_0$ .

The component  $\mathbf{p}$  is the **orthogonal projection** of the vector  $\mathbf{x}$  onto the subspace  $V_0$ . We have

$$\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V_0} \|\mathbf{x} - \mathbf{v}\|.$$

That is, the distance from  $\mathbf{x}$  to the subspace  $V_0$  is  $\|\mathbf{o}\|$ .



Let  $V$  be an inner product space. Let  $\mathbf{p}$  be the orthogonal projection of a vector  $\mathbf{x} \in V$  onto a finite-dimensional subspace  $V_0$ .

If  $V_0$  is a one-dimensional subspace spanned by a vector  $\mathbf{v}$  then  $\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$ .

If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is an orthogonal basis for  $V_0$  then

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \dots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n.$$

Indeed,  $\langle \mathbf{p}, \mathbf{v}_i \rangle = \sum_{j=1}^n \frac{\langle \mathbf{x}, \mathbf{v}_j \rangle}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{v}_i \rangle = \frac{\langle \mathbf{x}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle$

$$\implies \langle \mathbf{x} - \mathbf{p}, \mathbf{v}_i \rangle = 0 \implies \mathbf{x} - \mathbf{p} \perp \mathbf{v}_i \implies \mathbf{x} - \mathbf{p} \perp V_0.$$

## The Gram-Schmidt orthogonalization process

Let  $V$  be a vector space with an inner product. Suppose  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  is a basis for  $V$ . Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

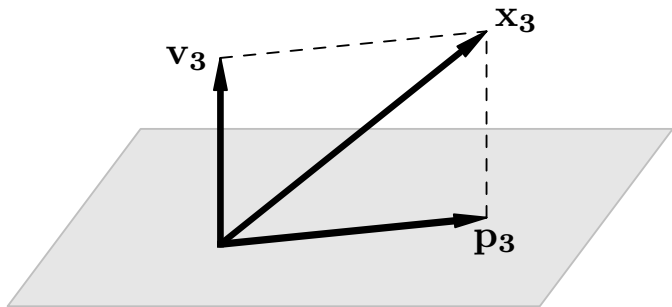
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

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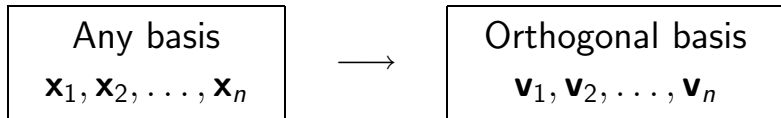
$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \dots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.$$

Then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is an orthogonal basis for  $V$ .



$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$$





*Properties of the Gram-Schmidt process:*

- $\mathbf{v}_k = \mathbf{x}_k - (\alpha_1 \mathbf{x}_1 + \dots + \alpha_{k-1} \mathbf{x}_{k-1})$ ,  $1 \leq k \leq n$ ;
- the span of  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is the same as the span of  $\mathbf{x}_1, \dots, \mathbf{x}_k$ ;
- $\mathbf{v}_k$  is orthogonal to  $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ ;
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$ , where  $\mathbf{p}_k$  is the orthogonal projection of the vector  $\mathbf{x}_k$  on the subspace spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ ;
- $\|\mathbf{v}_k\|$  is the distance from  $\mathbf{x}_k$  to the subspace spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ .

## Normalization

Let  $V$  be a vector space with an inner product.

Suppose  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is an orthogonal basis for  $V$ .

Let  $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ ,  $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$ ,  $\dots$ ,  $\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$ .

Then  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$  is an orthonormal basis for  $V$ .

**Theorem** Any finite-dimensional vector space with an inner product has an orthonormal basis.

*Remark.* An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

## Orthogonalization / Normalization

An alternative form of the Gram-Schmidt process combines orthogonalization with normalization.

Suppose  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  is a basis for an inner product space  $V$ . Let

$$\mathbf{v}_1 = \mathbf{x}_1, \quad \mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \langle \mathbf{x}_2, \mathbf{w}_1 \rangle \mathbf{w}_1, \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \langle \mathbf{x}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{x}_3, \mathbf{w}_2 \rangle \mathbf{w}_2, \quad \mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|},$$

.....

$$\mathbf{v}_n = \mathbf{x}_n - \langle \mathbf{x}_n, \mathbf{w}_1 \rangle \mathbf{w}_1 - \dots - \langle \mathbf{x}_n, \mathbf{w}_{n-1} \rangle \mathbf{w}_{n-1},$$

$$\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}.$$

Then  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$  is an orthonormal basis for  $V$ .

**Problem.** Let  $V_0$  be a subspace of dimension  $k$  in  $\mathbb{R}^n$ . Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$  be a basis for  $V_0$ .

(i) Find an orthogonal basis for  $V_0$ .

(ii) Extend it to an orthogonal basis for  $\mathbb{R}^n$ .

*Approach 1.* Extend  $\mathbf{x}_1, \dots, \mathbf{x}_k$  to a basis  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  for  $\mathbb{R}^n$ . Then apply the Gram-Schmidt process to the extended basis. We shall obtain an orthogonal basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  for  $\mathbb{R}^n$ . By construction,  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \text{Span}(\mathbf{x}_1, \dots, \mathbf{x}_k) = V_0$ . It follows that  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is a basis for  $V_0$ . Clearly, it is orthogonal.

*Approach 2.* First apply the Gram-Schmidt process to  $\mathbf{x}_1, \dots, \mathbf{x}_k$  and obtain an orthogonal basis  $\mathbf{v}_1, \dots, \mathbf{v}_k$  for  $V_0$ . Secondly, find a basis  $\mathbf{y}_1, \dots, \mathbf{y}_m$  for the orthogonal complement  $V_0^\perp$  and apply the Gram-Schmidt process to it obtaining an orthogonal basis  $\mathbf{u}_1, \dots, \mathbf{u}_m$  for  $V_0^\perp$ . Then  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$  is an orthogonal basis for  $\mathbb{R}^n$ .

**Problem.** Let  $\Pi$  be the plane in  $\mathbb{R}^3$  spanned by vectors  $\mathbf{x}_1 = (1, 2, 2)$  and  $\mathbf{x}_2 = (-1, 0, 2)$ .

- (i) Find an orthonormal basis for  $\Pi$ .
- (ii) Extend it to an orthonormal basis for  $\mathbb{R}^3$ .

$\mathbf{x}_1, \mathbf{x}_2$  is a basis for the plane  $\Pi$ . We can extend it to a basis for  $\mathbb{R}^3$  by adding one vector from the standard basis. For instance, vectors  $\mathbf{x}_1, \mathbf{x}_2$ , and  $\mathbf{x}_3 = (0, 0, 1)$  form a basis for  $\mathbb{R}^3$  because

$$\begin{vmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0.$$

Using the Gram-Schmidt process, we orthogonalize the basis  $\mathbf{x}_1 = (1, 2, 2)$ ,  $\mathbf{x}_2 = (-1, 0, 2)$ ,  $\mathbf{x}_3 = (0, 0, 1)$ :

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 2, 2),$$

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (-1, 0, 2) - \frac{3}{9}(1, 2, 2) \\ &= (-4/3, -2/3, 4/3),\end{aligned}$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \\ &= (0, 0, 1) - \frac{2}{9}(1, 2, 2) - \frac{4/3}{4}(-4/3, -2/3, 4/3) \\ &= (2/9, -2/9, 1/9).\end{aligned}$$

Now  $\mathbf{v}_1 = (1, 2, 2)$ ,  $\mathbf{v}_2 = (-4/3, -2/3, 4/3)$ ,  $\mathbf{v}_3 = (2/9, -2/9, 1/9)$  is an orthogonal basis for  $\mathbb{R}^3$  while  $\mathbf{v}_1, \mathbf{v}_2$  is an orthogonal basis for  $\Pi$ . It remains to normalize these vectors.

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 9 \implies \|\mathbf{v}_1\| = 3$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 4 \implies \|\mathbf{v}_2\| = 2$$

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = 1/9 \implies \|\mathbf{v}_3\| = 1/3$$

$$\mathbf{w}_1 = \mathbf{v}_1 / \|\mathbf{v}_1\| = (1/3, 2/3, 2/3) = \frac{1}{3}(1, 2, 2),$$

$$\mathbf{w}_2 = \mathbf{v}_2 / \|\mathbf{v}_2\| = (-2/3, -1/3, 2/3) = \frac{1}{3}(-2, -1, 2),$$

$$\mathbf{w}_3 = \mathbf{v}_3 / \|\mathbf{v}_3\| = (2/3, -2/3, 1/3) = \frac{1}{3}(2, -2, 1).$$

$\mathbf{w}_1, \mathbf{w}_2$  is an orthonormal basis for  $\Pi$ .

$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  is an orthonormal basis for  $\mathbb{R}^3$ .