

MATH 304

Linear Algebra

**Lecture 32:**

**Bases of eigenvectors.**

**Diagonalization.**

## Eigenvalues and eigenvectors of an operator

*Definition.* Let  $V$  be a vector space and  $L : V \rightarrow V$  be a linear operator. A number  $\lambda$  is called an **eigenvalue** of the operator  $L$  if  $L(\mathbf{v}) = \lambda\mathbf{v}$  for a nonzero vector  $\mathbf{v} \in V$ . The vector  $\mathbf{v}$  is called an **eigenvector** of  $L$  associated with the eigenvalue  $\lambda$ . (If  $V$  is a functional space then eigenvectors are also called **eigenfunctions**.)

If  $V = \mathbb{R}^n$  then the linear operator  $L$  is given by  $L(\mathbf{x}) = A\mathbf{x}$ , where  $A$  is an  $n \times n$  matrix.

In this case, eigenvalues and eigenvectors of the operator  $L$  are precisely eigenvalues and eigenvectors of the matrix  $A$ .

## Characteristic polynomial of an operator

Let  $L$  be a linear operator on a finite-dimensional vector space  $V$ . Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  be a basis for  $V$ . Let  $A$  be the matrix of  $L$  with respect to this basis.

*Definition.* The characteristic polynomial  $p(\lambda) = \det(A - \lambda I)$  of the matrix  $A$  is called the **characteristic polynomial** of the operator  $L$ .

Then eigenvalues of  $L$  are roots of its characteristic polynomial.

**Theorem.** The characteristic polynomial of the operator  $L$  is well defined. That is, it does not depend on the choice of a basis.

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*Proof:* Let  $B$  be the matrix of  $L$  with respect to a different basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . Then  $A = UBU^{-1}$ , where  $U$  is the transition matrix from the basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  to  $\mathbf{u}_1, \dots, \mathbf{u}_n$ . We have to show that  $\det(A - \lambda I) = \det(B - \lambda I)$  for all  $\lambda \in \mathbb{R}$ . We obtain

$$\begin{aligned} \det(A - \lambda I) &= \det(UBU^{-1} - \lambda I) \\ &= \det(UBU^{-1} - U(\lambda I)U^{-1}) = \det(U(B - \lambda I)U^{-1}) \\ &= \det(U) \det(B - \lambda I) \det(U^{-1}) = \det(B - \lambda I). \end{aligned}$$

## Basis of eigenvectors

**Theorem 1** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are eigenvectors of a linear operator  $L$  associated with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , then  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent.

**Theorem 2** Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be distinct eigenvalues of a linear operator  $L$ . For any  $1 \leq i \leq k$  let  $S_i$  be a basis for the eigenspace associated with the eigenvalue  $\lambda_i$ . Then the union  $S_1 \cup S_2 \cup \dots \cup S_k$  is a linearly independent set.

**Corollary** Let  $A$  be an  $n \times n$  matrix such that the characteristic equation  $\det(A - \lambda I) = 0$  has  $n$  distinct real roots. Then

- (i) there exists a basis for  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ ;
- (ii) all eigenspaces of  $A$  are one-dimensional.

## Diagonalization

Let  $L$  be a linear operator on a finite-dimensional vector space  $V$ . Then the following conditions are equivalent:

- the matrix of  $L$  with respect to some basis is diagonal;
- there exists a basis for  $V$  formed by eigenvectors of  $L$ .

The operator  $L$  is **diagonalizable** if it satisfies these conditions.

Let  $A$  be an  $n \times n$  matrix. Then the following conditions are equivalent:

- $A$  is the matrix of a diagonalizable operator;
- $A$  is similar to a diagonal matrix, i.e., it is represented as  $A = UBU^{-1}$ , where the matrix  $B$  is diagonal;
- there exists a basis for  $\mathbb{R}^n$  formed by eigenvectors of  $A$ .

The matrix  $A$  is **diagonalizable** if it satisfies these conditions. Otherwise  $A$  is called **defective**.

*Example.*  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

- The matrix  $A$  has two eigenvalues: 1 and 3.
- The eigenspace of  $A$  associated with the eigenvalue 1 is the line spanned by  $\mathbf{v}_1 = (-1, 1)$ .
- The eigenspace of  $A$  associated with the eigenvalue 3 is the line spanned by  $\mathbf{v}_2 = (1, 1)$ .
- Eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  form a basis for  $\mathbb{R}^2$ .

Thus the matrix  $A$  is diagonalizable. Namely,  
 $A = UBU^{-1}$ , where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

*Example.*  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$

- The matrix  $A$  has two eigenvalues: 0 and 2.
- The eigenspace corresponding to 0 is spanned by  $\mathbf{v}_1 = (-1, 1, 0).$
- The eigenspace corresponding to 2 is spanned by  $\mathbf{v}_2 = (1, 1, 0)$  and  $\mathbf{v}_3 = (-1, 0, 1).$
- Eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  form a basis for  $\mathbb{R}^3.$

Thus the matrix  $A$  is diagonalizable. Namely,  
 $A = UBU^{-1},$  where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are **two obstructions** to existence of a basis consisting of eigenvectors. They are illustrated by the following examples.

*Example 1.*  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

$\det(A - \lambda I) = (\lambda - 1)^2$ . Hence  $\lambda = 1$  is the only eigenvalue. The associated eigenspace is the line  $t(1, 0)$ .

*Example 2.*  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

$\det(A - \lambda I) = \lambda^2 + 1$ .

$\implies$  no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

**Problem.** Diagonalize the matrix  $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ .

We need to find a diagonal matrix  $B$  and an invertible matrix  $U$  such that  $A = UBU^{-1}$ .

Suppose that  $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$  is a basis for  $\mathbb{R}^2$  formed by eigenvectors of  $A$ , i.e.,  $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$  for some  $\lambda_i \in \mathbb{R}$ . Then we can take

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

Note that  $U$  is the transition matrix from the basis  $\mathbf{v}_1, \mathbf{v}_2$  to the standard basis.

**Problem.** Diagonalize the matrix  $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ .

Characteristic equation of  $A$ :  $\begin{vmatrix} 4 - \lambda & 3 \\ 0 & 1 - \lambda \end{vmatrix} = 0$ .

$$(4 - \lambda)(1 - \lambda) = 0 \implies \lambda_1 = 4, \lambda_2 = 1.$$

Associated eigenvectors:  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

Thus  $A = UBU^{-1}$ , where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

**Problem.** Let  $A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$ . Find  $A^5$ .

We know that  $A = UBU^{-1}$ , where

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then  $A^5 = UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}UBU^{-1}$

$$= UB^5U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1024 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1024 & 1023 \\ 0 & 1 \end{pmatrix}.$$