MATH 304

Linear Algebra

Lecture 40:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.

- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process
- Orthogonal polynomials

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization
- Matrix exponentials
- Complex eigenvalues and eigenvectors
- Orthogonal matrices
- Rotations in space

Bases of eigenvectors

Let A be an $n \times n$ matrix with real entries.

- A has n distinct real eigenvalues $\Longrightarrow$ a basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has complex eigenvalues $\Longrightarrow$ no basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has n distinct complex eigenvalues $\Longrightarrow$ a basis for $\mathbb{C}^n$ formed by eigenvectors of A
- A has multiple eigenvalues $\Longrightarrow$ further information is needed
- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of A $\Longleftrightarrow$ A is symmetric: $A^T = A$

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,

(b) a basis of eigenvectors for $\mathbb{C}^2$,

(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ (a),(b),(c): yes

$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (a),(b),(c): no

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ (a),(b): yes (c): no

$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (b): yes (a),(c): no

Problem. Let V be the vector space spanned by functions $f_1(x) = x\sin x$, $f_2(x) = x\cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$f_1'(x) = (x\sin x)' = x\cos x + \sin x = f_2(x) + f_3(x),$$
$$f_2'(x) = (x\cos x)' = -x\sin x + \cos x$$
$$= -f_1(x) + f_4(x),$$
$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$
$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 1) + (\lambda^2 + 1) = (\lambda^2 + 1)^2.$$

The eigenvalues are i and $-i$, both of multiplicity 2.

Complex eigenvalues $\implies$ A is not diagonalizable in $\mathbb{R}^4$

If A is diagonalizable in $\mathbb{C}^4$ then $A = UXU^{-1}$, where U is an invertible matrix with complex entries and

$$X = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

This would imply that $A^2 = UX^2U^{-1}$. But $X^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, the matrix A is not diagonalizable in $\mathbb{C}^4$.

Problem. Let R denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $R(\mathbf{e}_1) = \mathbf{e}_2$, $R(\mathbf{e}_2) = \mathbf{e}_3$, $R(\mathbf{e}_3) = -\mathbf{e}_1$. Is R a rotation about an axis? Is R a reflection in a plane?

The matrix of R relative to the standard basis is

$$M = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

M is orthogonal since columns form an orthonormal basis for $\mathbb{R}^3$. According to the classification of the 3×3 orthogonal matrices, R is either a rotation about an axis, or a reflection in a plane, or the composition of a rotation about an axis with the reflection in the plane orthogonal to the axis.

R is not a rotation since $\det M \neq 1$.
R is not a reflection since $M^2 \neq I$.

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2011} (L applied 2011 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3.$$

In particular, $L(\mathbf{e}_1) = -\frac{4}{5}\mathbf{e}_2$, $L(\mathbf{e}_2) = \frac{4}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_3$, $L(\mathbf{e}_3) = -\frac{3}{5}\mathbf{e}_2$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

$$B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2011} is B^{2011}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2011} = UD^{2011}U^{-1}$. We have that $D^{2011} =$
$= \mathrm{diag}\big(0, i^{2011}, (-i)^{2011}\big) = \mathrm{diag}(0, -i, i) = -D$.
Hence

$$B^{2011} = U(-D)U^{-1} = -B = \begin{pmatrix} 0 & -4/5 & 0 \\ 4/5 & 0 & 3/5 \\ 0 & -3/5 & 0 \end{pmatrix}.$$