MATH 304

Linear Algebra

Lecture 2:
Gaussian elimination.

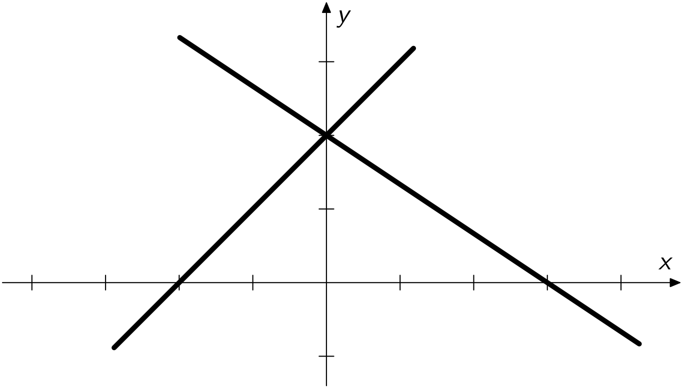
System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

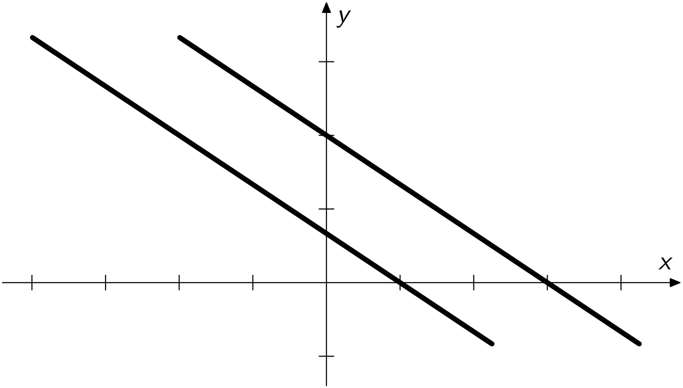
Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

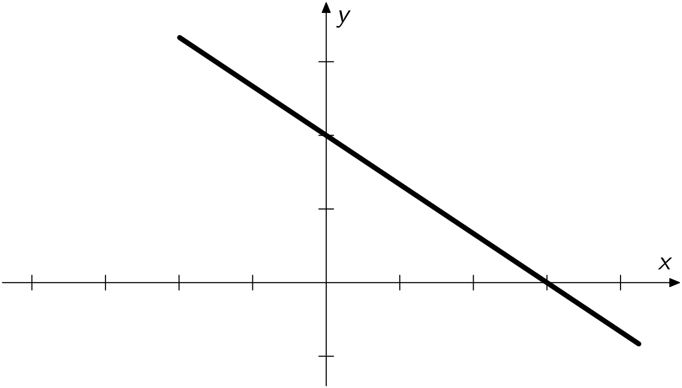


$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \qquad x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system
(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = -2 \implies x = y - 2$$

$$2x + 3y = 6 \implies 2(y - 2) + 3y = 6$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 2 \implies x = y - 2 = 0$$

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = y + 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Eliminate x from the 2nd and 3rd equations:

$$\begin{cases} x = y + 2 \\ 2(y + 2) - y - z = 3 \\ (y + 2) + y + z = 6 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Now the 2nd and 3rd equations form the system of two linear equations in two variables.

Solve the 2nd equation for y:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2y + z = 4 \end{cases}$$

Eliminate y from the 3rd equation:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2(z - 1) + z = 4 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

The elimination is completed. Now the system is easily solved by back substitution.

That is, we find z from the 3rd equation, then substitute it in the 2nd equation and find y, then substitute y and z in the 1st equation and find x.

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y &= 2 \\ 2x - y - z &= 3 \\ x + y + z &= 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Eliminate x from the 3rd equations:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -(-y + 2z + 1) + 4y - 3z = 14 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Solve the 2nd equation for y:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5y - 5z = 15 \end{cases}$$

Eliminate y from the 3rd equations:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5(z + 3) - 5z = 15 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 0 = 0 \end{cases}$$

The elimination is completed.

The last equation is actually $0z = 0$. Hence z is a *free variable*, i.e., it can be assigned an arbitrary value. Then y and x are found by back substitution.

$z = t$, a parameter;
$y = z + 3 = t + 3$;
$x = -y + 2z + 1 = -(t + 3) + 2t + 1 = t - 2.$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

General solution:

$$(x, y, z) = (t - 2, t + 3, t), \ \ t \in \mathbb{R}.$$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1)$.

The set of all solutions is a straight line in $\mathbb{R}^3$ passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Theorem (i) Applying elementary operations to a system of linear equations does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation.

Operation 1: multiply the ith equation by $r \neq 0$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Longrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

To undo the operation, multiply the ith equation by r^{-1}.

Operation 2: add r times the ith equation to the jth equation.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases} \implies$$

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ (a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, add $-r$ times the ith equation to the jth equation.

Operation 3: interchange the *i*th and *j*th equations.

$$\begin{cases} \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \end{cases}$$

$$\Longrightarrow \begin{cases} \cdots\cdots\cdots\cdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \cdots\cdots\cdots\cdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \cdots\cdots\cdots\cdots \end{cases}$$

To undo the operation, apply it once more.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Add -2 times the 1st equation to the 2nd equation:

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ x + y + z = 6 \end{cases} \qquad \boxed{R2 := R2 - 2 * R1}$$

Add -1 times the 1st equation to the 3rd equation:

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Add -2 times the 2nd equation to the 3rd equation:

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ 3z = 6 \end{cases}$$

The elimination is completed, and we can solve the system by back substitution. However we can as well proceed with elementary operations.

Multiply the 3rd equation by $1/3$:

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ z = 2 \end{cases}$$

Add the 3rd equation to the 2nd equation:

$$\begin{cases} x - y = 2 \\ y = 1 \\ z = 2 \end{cases}$$

Add the 2nd equation to the 1st equation:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y &= 2 \\ 2x - y - z &= 3 \\ x + y + z &= 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases}$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x \quad\quad - z = -2 \\ \quad\quad y - z = 3 \\ \quad\quad\quad\quad 0 = 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that $\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases}$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \ \ t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

The set of all solutions is a straight line in $\mathbb{R}^3$ passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).