MATH 304

Linear Algebra

Lecture 3:
Row echelon form.
Gauss-Jordan reduction.

Gaussian elimination

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution. Both parts can be done by applying a finite number of **elementary operations**:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \to \begin{cases} x - y = 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

$$\to \begin{cases} x - y = 2 \\ y - z = -1 \\ 3z = 6 \end{cases} \to \begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases}$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x \qquad\; - z = -2 \\ \quad\; y - z = 3 \\ \qquad\quad\;\; 0 = 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that $\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases}$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \quad t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $(\sqrt{2}, 0, -\sqrt{3}, 5)$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions $=$ **(# of rows)** $\times$ **(# of columns)**

n-by-n: **square matrix**

n-by-1: **column vector**

1-by-n: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

Elementary operations for systems of linear equations correspond to **elementary row operations** for augmented matrices:
(1) to multiply a row by a nonzero scalar;
(2) to add the ith row multiplied by some $r \in \mathbb{R}$ to the jth row;
(3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1}\ a_{i2}\ \dots\ a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Row echelon form

Definition. **Leading entry** of a matrix is the first nonzero entry in a row.

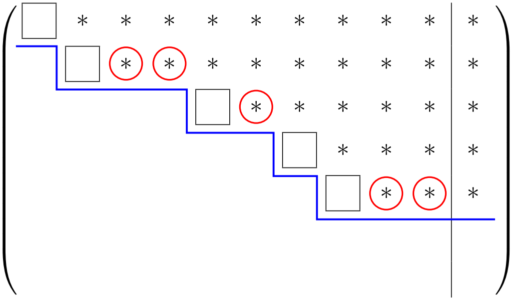
The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

$$\left(\begin{array}{cccccccccc|c}
1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\
0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

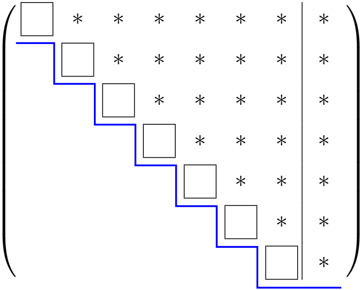
Row echelon form

General augmented matrix in row echelon form:



- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.

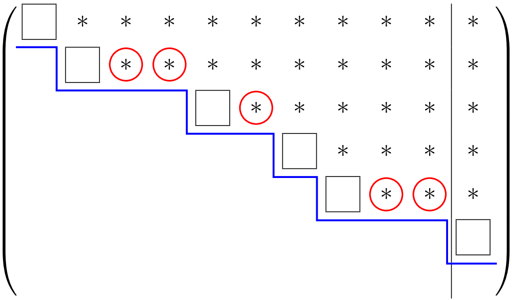
Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:



- no zero rows;
- no free variables.

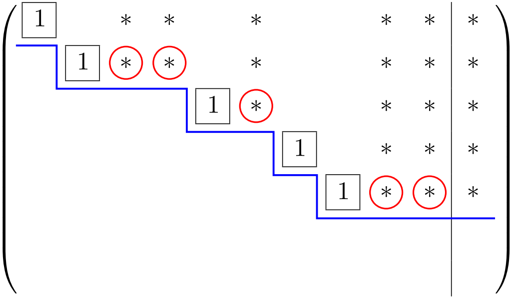
Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.



Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:



- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.

Example.

$$\begin{cases} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \qquad \left(\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y = 2 \\ y - z = -1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{rrr|r} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right)$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

New example. $\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$

Variables: x_1, x_2, x_3, x_4.

Augmented matrix: $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array}\right)$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

The leading variables are x_1 and x_2; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$

$$= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$$