

MATH 304

Linear Algebra

**Lecture 10:**

**Evaluation of determinants.**

**Cramer's rule.**

## Determinants: definition in low dimensions

*Definition.*  $\det(a) = a$ ,  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ ,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

## Properties of determinants

*Determinants and elementary row operations:*

- if a row of a matrix is multiplied by a scalar  $r$ , the determinant is also multiplied by  $r$ ;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

## Properties of determinants

*Tests for singularity:*

- if a matrix  $A$  has a zero row then  $\det A = 0$ ;
- if a matrix  $A$  has two identical rows then  $\det A = 0$ ;
- if a matrix  $A$  has two proportional rows then  $\det A = 0$ ;
- if a matrix  $A$  is not invertible then  $\det A = 0$ .

## Properties of determinants

*Special matrices:*

- $\det I = 1$ ;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

## Properties of determinants

*Determinant of the transpose:*

- If  $A$  is a square matrix then  $\det A^T = \det A$ .

*Columns vs. rows:*

- if one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar;
- adding a scalar multiple of one column to another does not change the determinant;
- interchanging two columns of a matrix changes the sign of its determinant;
- if a matrix  $A$  has a zero column or two proportional columns then  $\det A = 0$ .

## Row and column expansions

Given an  $n \times n$  matrix  $A = (a_{ij})$ , let  $M_{ij}$  denote the  $(n-1) \times (n-1)$  submatrix obtained by deleting the  $i$ th row and the  $j$ th column of  $A$ .

**Theorem** For any  $1 \leq k, m \leq n$  we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

*(expansion by  $k$ th row)*

$$\det A = \sum_{i=1}^n (-1)^{i+m} a_{im} \det M_{im}.$$

*(expansion by  $m$ th column)*

## Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



## Evaluation of determinants

Example.  $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$ .

First let's do some row reduction.

Add  $-4$  times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add  $-7$  times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\begin{aligned} \det B &= \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix} \\ &= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12. \end{aligned}$$

*Example.*  $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$ ,  $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add  $-2$  times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

**Problem.** For what values of  $a$  will the following system have a unique solution?

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add  $-2$  times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix} = -(-2 - 2a) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence  $\det A = -(-2 - 2a) \cdot 3 = 6(1 + a)$ .

Thus  $A$  is invertible if and only if  $a \neq -1$ .

## More properties of determinants

*Determinants and matrix multiplication:*

- if  $A$  and  $B$  are  $n \times n$  matrices then
$$\det(AB) = \det A \cdot \det B;$$
- if  $A$  and  $B$  are  $n \times n$  matrices then
$$\det(AB) = \det(BA);$$
- if  $A$  is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

*Determinants and scalar multiplication:*

- if  $A$  is an  $n \times n$  matrix and  $r \in \mathbb{R}$  then
$$\det(rA) = r^n \det A.$$

## Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

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$$\det X = (-1) \cdot 2 \cdot (-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6 \cdot 3 = 18, \quad \det(YX) = 3 \cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$$

$$\det(-3X^TXY^{-4}) = (-3)^3 \cdot 6 \cdot 6 \cdot 3^{-4} = -12.$$



## Cramer's rule

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b}$$

**Theorem** Assume that the matrix  $A$  is invertible. Then the only solution of the system is given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, 2, \dots, n,$$

where the matrix  $A_i$  is obtained by substituting the vector  $\mathbf{b}$  for the  $i$ th column of  $A$ .

## Determinants and the inverse matrix

Given an  $n \times n$  matrix  $A = (a_{ij})$ , let  $M_{ij}$  denote the  $(n-1) \times (n-1)$  submatrix obtained by deleting the  $i$ th row and the  $j$ th column of  $A$ . The **cofactor matrix** of  $A$  is an  $n \times n$  matrix  $\tilde{A} = (\alpha_{ij})$  defined by  $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$ .

**Theorem**  $\tilde{A}^T A = A \tilde{A}^T = (\det A)I$ .

*Sketch of the proof:*  $A \tilde{A}^T = (\det A)I$  means that

$$\sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^n (-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$

Indeed, the 1st equality is the expansion of  $\det A$  by the  $k$ th row. The 2nd equality is an analogous expansion of  $\det B$ , where the matrix  $B$  is obtained from  $A$  by replacing its  $k$ th row with a copy of the  $m$ th row (clearly,  $\det B = 0$ ).

$\tilde{A}^T A = (\det A)I$  is verified similarly, using column expansions.

**Corollary** If  $\det A \neq 0$  then  $A^{-1} = (\det A)^{-1} \tilde{A}^T$ .