MATH 304

Linear Algebra

Lecture 11:
Vector spaces.

Linear operations on vectors

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

Scalar multiple: $r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Negative of a vector: $-\mathbf{y} = (-y_1, -y_2, \ldots, -y_n)$

Vector difference:
$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$

Properties of linear operations

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$$

$$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$$

$$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$$

$$(rs)\mathbf{x} = r(s\mathbf{x})$$

$$1\mathbf{x} = \mathbf{x}$$

$$0\mathbf{x} = \mathbf{0}$$

$$(-1)\mathbf{x} = -\mathbf{x}$$

Linear operations on matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices, and $r \in \mathbb{R}$ be a scalar.

Matrix sum: $A + B = (a_{ij} + b_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Scalar multiple: $rA = (ra_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Zero matrix O: all entries are zeros

Negative of a matrix: $-A = (-a_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

Matrix difference: $A - B = (a_{ij} - b_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

As far as the linear operations are concerned, the $m \times n$ matrices have the same properties as mn-dimensional vectors.

Vector space: informal description

Vector space = *linear space* = a set V of objects (called *vectors*) that can be added and scaled.

That is, for any $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$ expressions

$$\boxed{\mathbf{u} + \mathbf{v}} \text{ and } \boxed{r\mathbf{u}}$$

should make sense.

Certain restrictions apply. For instance,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$2\mathbf{u} + 3\mathbf{u} = 5\mathbf{u}.$$

That is, addition and scalar multiplication in V should be like those of n-dimensional vectors.

Vector space: definition

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication (brief)

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

A3. $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$

A4. $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$

A5. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$

A6. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$

A7. $(rs)\mathbf{x} = r(s\mathbf{x})$

A8. $1\mathbf{x} = \mathbf{x}$

Properties of addition and scalar multiplication (detailed)

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$.

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

A4. For any $\mathbf{x} \in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

A5. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ for all $r \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$.

A6. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$.

A7. $(rs)\mathbf{x} = r(s\mathbf{x})$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$.

A8. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

- Associativity of addition implies that a multiple sum $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$ is well defined for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.

- **Subtraction** in V is defined as follows: $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y})$.

- Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties A1–A8 are easy to verify.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$

For any $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots) \in \mathbb{R}^\infty$ and $r \in \mathbb{R}$ let $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots)$, $r\mathbf{x} = (rx_1, rx_2, \dots)$. Then $\mathbf{0} = (0, 0, \dots)$ and $-\mathbf{x} = (-x_1, -x_2, \dots)$.

- $\{\mathbf{0}\}$: the trivial vector space

$\mathbf{0} + \mathbf{0} = \mathbf{0}$, $r\mathbf{0} = \mathbf{0}$, $-\mathbf{0} = \mathbf{0}$.

Functional vector spaces

- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$

Given functions $f, g \in F(\mathbb{R})$ and a scalar $r \in \mathbb{R}$, let $(f+g)(x) = f(x) + g(x)$ and $(rf)(x) = rf(x)$ for all $x \in \mathbb{R}$. Zero vector: $o(x) = 0$. Negative: $(-f)(x) = -f(x)$.

- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

Linear operations are inherited from $F(\mathbb{R})$. We only need to check that $f, g \in C(\mathbb{R}) \implies f+g, rf \in C(\mathbb{R})$, the zero function is continuous, and $f \in C(\mathbb{R}) \implies -f \in C(\mathbb{R})$.

- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$

Some general observations

- The zero vector is unique.

Suppose $\mathbf{z}_1$ and $\mathbf{z}_2$ are zero vectors. Then $\mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_2$ since $\mathbf{z}_1$ is a zero vector and $\mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_1$ since $\mathbf{z}_2$ is a zero vector. Hence $\mathbf{z}_1 = \mathbf{z}_2$.

- For any $\mathbf{x} \in V$, the negative $-\mathbf{x}$ is unique.

Suppose $\mathbf{y}$ and $\mathbf{y}'$ are both negatives of $\mathbf{x}$. Let us compute the sum $\mathbf{y}' + \mathbf{x} + \mathbf{y}$ in two ways:

$$(\mathbf{y}' + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y},$$
$$\mathbf{y}' + (\mathbf{x} + \mathbf{y}) = \mathbf{y}' + \mathbf{0} = \mathbf{y}'.$$

By associativity of the vector addition, $\mathbf{y} = \mathbf{y}'$.

Some general observations

- (cancellation law) $\mathbf{x}+\mathbf{y}=\mathbf{x}'+\mathbf{y}$ implies $\mathbf{x}=\mathbf{x}'$ for any $\mathbf{x},\mathbf{x}',\mathbf{y}\in V$.

If $\mathbf{x}+\mathbf{y}=\mathbf{x}'+\mathbf{y}$ then $(\mathbf{x}+\mathbf{y})+(-\mathbf{y})=(\mathbf{x}'+\mathbf{y})+(-\mathbf{y})$. By associativity, $(\mathbf{x}+\mathbf{y})+(-\mathbf{y})=\mathbf{x}+(\mathbf{y}+(-\mathbf{y}))=\mathbf{x}+\mathbf{0}=\mathbf{x}$ and $(\mathbf{x}'+\mathbf{y})+(-\mathbf{y})=\mathbf{x}'+(\mathbf{y}+(-\mathbf{y}))=\mathbf{x}'+\mathbf{0}=\mathbf{x}'$. Hence $\mathbf{x}=\mathbf{x}'$.

- $0\mathbf{x}=\mathbf{0}$ for any $\mathbf{x}\in V$.

Indeed, $0\mathbf{x}+\mathbf{x}=0\mathbf{x}+1\mathbf{x}=(0+1)\mathbf{x}=1\mathbf{x}=\mathbf{x}=\mathbf{0}+\mathbf{x}$. By the cancellation law, $0\mathbf{x}=\mathbf{0}$.

- $(-1)\mathbf{x}=-\mathbf{x}$ for any $\mathbf{x}\in V$.

Indeed, $\mathbf{x}+(-1)\mathbf{x}=(-1)\mathbf{x}+\mathbf{x}=(-1)\mathbf{x}+1\mathbf{x}=(-1+1)\mathbf{x}$ $=0\mathbf{x}=\mathbf{0}$.

Counterexample: dumb scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{0}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

$$\begin{array}{lll}
\text{A5. } r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} & \iff & \mathbf{0} = \mathbf{0} + \mathbf{0} \\
\text{A6. } (r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} & \iff & \mathbf{0} = \mathbf{0} + \mathbf{0} \\
\text{A7. } (rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) & \iff & \mathbf{0} = \mathbf{0} \\
\text{A8. } 1 \odot \mathbf{x} = \mathbf{x} & \iff & \mathbf{0} = \mathbf{x}
\end{array}$$

A8 is the only property that fails. As a consequence, property A8 does not follow from properties A1–A7.

Counterexample: lazy scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{x} = \mathbf{x}} \quad \text{for any } \mathbf{x} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

A5. $r \odot (\mathbf{x} + \mathbf{y}) = r \odot \mathbf{x} + r \odot \mathbf{y} \iff \mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}$

A6. $(r + s) \odot \mathbf{x} = r \odot \mathbf{x} + s \odot \mathbf{x} \iff \mathbf{x} = \mathbf{x} + \mathbf{x}$

A7. $(rs) \odot \mathbf{x} = r \odot (s \odot \mathbf{x}) \iff \mathbf{x} = \mathbf{x}$

A8. $1 \odot \mathbf{x} = \mathbf{x} \iff \mathbf{x} = \mathbf{x}$

The only property that fails is A6.

Weird example

Consider the set $V = \mathbb{R}_+$ of positive numbers with a nonstandard addition and scalar multiplication:

$$\boxed{x \oplus y = xy} \quad \text{for any } x, y \in \mathbb{R}_+.$$

$$\boxed{r \odot x = x^r} \quad \text{for any } x \in \mathbb{R}_+ \text{ and } r \in \mathbb{R}.$$

A1. $x \oplus y = y \oplus x \quad \iff \quad xy = yx$

A2. $(x \oplus y) \oplus z = x \oplus (y \oplus z) \quad \iff \quad (xy)z = x(yz)$

A3. $x \oplus \zeta = \zeta \oplus x = x \quad \iff \quad x\zeta = \zeta x = x$ (holds for $\zeta = 1$)

A4. $x \oplus \eta = \eta \oplus x = 1 \quad \iff \quad x\eta = \eta x = 1$ (holds for $\eta = x^{-1}$)

A5. $r \odot (x \oplus y) = (r \odot x) \oplus (r \odot y) \quad \iff \quad (xy)^r = x^r y^r$

A6. $(r + s) \odot x = (r \odot x) \oplus (s \odot x) \quad \iff \quad x^{r+s} = x^r x^s$

A7. $(rs) \odot x = r \odot (s \odot x) \quad \iff \quad x^{rs} = (x^s)^r$

A8. $1 \odot x = x \quad \iff \quad x^1 = x$