MATH 304

Linear Algebra

Lecture 14: Linear independence.

Spanning set

Let S be a subset of a vector space V.

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S. If S is not empty then $W = \mathrm{Span}(S)$ consists of all linear combinations $\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$ and $r_1, \dots, r_k \in \mathbb{R}$.

We say that the set S **spans** the subspace W or that S is a **spanning set** for W.

Remarks. • If S_1 is a spanning set for a vector space V and $S_1 \subset S_2 \subset V$, then S_2 is also a spanning set for V.

• If $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a spanning set for V and $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x, y, z) = \mathbf{0}$$
$$\implies x = y = z = 0$$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O$$
$$\implies a = b = c = d = 0$$

Examples of linear independence

- Polynomials $1, x, x^2, \dots, x^n$.

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ identically
$\implies a_i = 0$ for $0 \leq i \leq n$

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x-1)^2$.

$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = a_1 + a_2(x-1) + a_3(x-1)^2 =$
$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3 x^2$.

Hence $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ identically
$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$
$\implies a_1 = a_2 = a_3 = 0$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{v}_3 = (4, -7, 3)$. Determine whether vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular. We obtain that $\det A = 0$.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: **(i)** $\Longrightarrow$ **(ii)** Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\tfrac{r_1}{r_i}\mathbf{v}_1 - \cdots - \tfrac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \tfrac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \tfrac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ **(i)** Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

Proof: Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \ldots, a_{nj})$ for $j = 1, 2, \ldots, m$. Then the vector equality $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \cdots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \cdots + a_{2m}t_m = 0, \\ \qquad\qquad \cdots\cdots\cdots \\ a_{n1}t_1 + a_{n2}t_2 + \cdots + a_{nm}t_m = 0. \end{cases}$$

Note that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are columns of the matrix (a_{ij}). The number of leading entries in the row echelon form is at most n. If $m > n$ then there are free variables, therefore the zero solution is not unique.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Two vectors are linearly dependent if and only if they are parallel. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in $\mathbb{R}^3$ are always linearly dependent.

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Problem. Let $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Determine whether matrices A, A^2, and A^3 are linearly independent.

We have $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The task is to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1 A + r_2 A^2 + r_3 A^3 = O$.

This matrix equation is equivalent to a system

$$\begin{cases} -r_1 + 0r_2 + r_3 = 0 \\ r_1 - r_2 + 0r_3 = 0 \\ -r_1 + r_2 + 0r_3 = 0 \\ 0r_1 - r_2 + r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is $A + A^2 + A^3 = O$).

More facts on linear independence

Let S_0 and S be subsets of a vector space V.

- If $S_0 \subset S$ and S is linearly independent, then so is S_0.
- If $S_0 \subset S$ and S_0 is linearly dependent, then so is S.
- If S is linearly independent in V and V is a subspace of W, then S is linearly independent in W.
- The empty set is linearly independent.
- Any set containing $\mathbf{0}$ is linearly dependent.
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If S_0 is linearly independent and $\mathbf{v}_0 \in V \setminus S_0$ then $S_0 \cup \{\mathbf{v}_0\}$ is linearly independent if and only if $\mathbf{v}_0 \notin \mathrm{Span}(S_0)$.