MATH 304

Linear Algebra

Lecture 15:
Wronskian.
The Vandermonde determinant.
Basis of a vector space.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.

Some facts on linear independence

Let S_0 and S be subsets of a vector space V.

- If $S_0 \subset S$ and S is linearly independent, then so is S_0.
- If $S_0 \subset S$ and S_0 is linearly dependent, then so is S.
- If S is linearly independent in V and V is a subspace of W, then S is linearly independent in W.
- The empty set is linearly independent.
- Any set containing $\mathbf{0}$ is linearly dependent.
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If S_0 is linearly independent and $\mathbf{v}_0 \in V \setminus S_0$ then $S_0 \cup \{\mathbf{v}_0\}$ is linearly independent if and only if $\mathbf{v}_0 \notin \mathrm{Span}(S_0)$.

Problem. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$\begin{aligned} ae^x + be^{2x} + ce^{3x} &= 0, \\ ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ ae^x + 4be^{2x} + 9ce^{3x} &= 0. \end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\det A(x) = e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0.$$

Since the matrix $A(x)$ is invertible, we obtain

$$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$$

Wronskian

Let $f_1, f_2, \ldots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \ldots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \ldots, f_n$ are linearly independent in $C[a, b]$.

Theorem Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

$$W[e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}](x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_k x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \cdots & \lambda_k e^{\lambda_k x} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} e^{\lambda_1 x} & \lambda_2^{k-1} e^{\lambda_2 x} & \cdots & \lambda_k^{k-1} e^{\lambda_k x} \end{vmatrix}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_k)x} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix}.$$

The Vandermonde determinant

Definition. The **Vandermonde determinant** is the determinant of the following matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Equivalently, $V = (a_{ij})_{1 \le i,j \le n}$, where $a_{ij} = x_i^{j-1}$.

Examples.

- $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1.$

- $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & x_2^2 - x_1x_2 \\ 1 & x_3 & x_3^2 - x_1x_3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1x_2 \\ 1 & x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_2^2 - x_1x_2 \\ x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix}$$

$$= (x_2 - x_1) \begin{vmatrix} 1 & x_2 \\ x_3 - x_1 & x_3^2 - x_1x_3 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

Theorem

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Corollary The Vandermonde determinant is not equal to 0 if and only if the numbers $x_1, x_2, \ldots, x_n$ are distinct.

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers.

Theorem For any $b_1, b_2, \ldots, b_n \in \mathbb{R}$ there exists a unique polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ of degree less than n such that $p(x_i) = b_i$, $1 \leq i \leq n$.

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} = b_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} = b_2 \\ \cdots\cdots\cdots\cdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} = b_n \end{cases}$$

$a_0, a_1, \ldots, a_{n-1}$ are unknowns. The coefficient matrix is the Vandermonde matrix.

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V.
"Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k,$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \ldots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = r_1'\mathbf{v}_1 + r_2'\mathbf{v}_2 + \cdots + r_k'\mathbf{v}_k$$
$$\Longrightarrow \quad (r_1 - r_1')\mathbf{v}_1 + (r_2 - r_2')\mathbf{v}_2 + \cdots + (r_k - r_k')\mathbf{v}_k = \mathbf{0}$$
$$\Longrightarrow \quad r_1 - r_1' = r_2 - r_2' = \ldots = r_k - r_k' = 0$$

Examples. • Standard basis for $\mathbb{R}^n$:
$\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$,…,
$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.
Indeed, $(x_1, x_2, \dots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

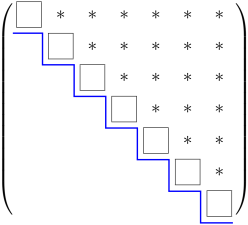
Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

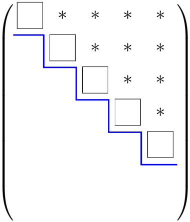
That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A.

- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span $\mathbb{R}^n$ if the row echelon form of A has no zero rows.*

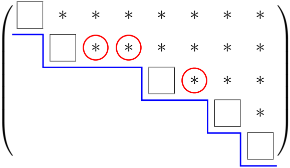
- *Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).*



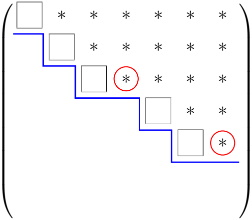
spanning
linear independence



no spanning
linear independence



spanning
no linear independence



no spanning
no linear independence

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.