

MATH 304
Linear Algebra

Lecture 21:
Properties of linear transformations.
Range and kernel.
General linear equations.

Linear transformation

Definition. Given vector spaces V_1 and V_2 , a mapping $L : V_1 \rightarrow V_2$ is **linear** if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

$$L(r\mathbf{x}) = rL(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear transformations

Let $L : V_1 \rightarrow V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$
for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{R}$.

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2),$$

$$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) = \\ = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + r_3L(\mathbf{v}_3), \text{ and so on.}$$

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2 , respectively.

$$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2.$$

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v}).$$

Examples of linear mappings

- *Scaling* $L : V \rightarrow V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{R}$.

- *Dot product with a fixed vector*

$$\ell : \mathbb{R}^n \rightarrow \mathbb{R}, \ell(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^n.$$

- *Cross product with a fixed vector*

$$L : \mathbb{R}^3 \rightarrow \mathbb{R}^3, L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \text{ where } \mathbf{v}_0 \in \mathbb{R}^3.$$

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m \times n$ matrix and all vectors are column vectors.

- *Coordinate mapping*

$L : V \rightarrow \mathbb{R}^n$, $L(\mathbf{v}) =$ coordinates of \mathbf{v} relative to an ordered basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ for the vector space V .

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$$\ell : F(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = f(a), \quad \text{where } a \in \mathbb{R}.$$

- *Multiplication by a fixed function*

$$L : F(\mathbb{R}) \rightarrow F(\mathbb{R}), \quad L(f) = gf, \quad \text{where } g \in F(\mathbb{R}).$$

- *Differentiation* $D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad L(f) = f'.$

- *Integration over a finite interval*

$$\ell : C(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = \int_a^b f(x) dx, \quad \text{where}$$

$a, b \in \mathbb{R}, \quad a < b.$

More properties of linear mappings

- If a linear mapping $L : V \rightarrow W$ is invertible then the inverse mapping $L^{-1} : W \rightarrow V$ is also linear.
- If $L : V \rightarrow W$ and $M : W \rightarrow X$ are linear mappings then the composition $M \circ L : V \rightarrow X$ is also linear.
- If $L_1 : V \rightarrow W$ and $L_2 : V \rightarrow W$ are linear mappings then the sum $L_1 + L_2$ is also linear.

Linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on \mathbb{R} .

That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Examples. $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \rightarrow \mathcal{M}_{n,m}(\mathbb{R}), \alpha(A) = A^T.$

$$\alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^T = A^T + B^T.$$

$$\alpha(rA) = r\alpha(A) \iff (rA)^T = rA^T.$$

Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathbb{R}, \beta(A) = \det A.$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

We have $\det(A) = \det(B) = 0$ while $\det(A + B) = 1.$

Hence $\beta(A + B) \neq \beta(A) + \beta(B)$ so that β is not linear.

Range and kernel

Let V, W be vector spaces and $L : V \rightarrow W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $L(V)$.

The **kernel** of L , denoted $\ker L$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem (i) The range of L is a subspace of W .
(ii) The kernel of L is a subspace of V .

Example. $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The kernel $\ker(L)$ is the nullspace of the matrix.

$$L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The range $L(\mathbb{R}^3)$ is the column space of the matrix.

Example. $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The range of L is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $L(\mathbb{R}^3)$ is the plane spanned by $(1, 1, 1)$ and $(0, 1, 0)$.

To find $\ker(L)$, we apply row reduction to the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \ker(L)$ if $x - z = y = 0$.

It follows that $\ker(L)$ is the line spanned by $(1, 0, 1)$.

Example. $L: C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$, $L(u) = u''' - 2u'' + u'$.

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), & x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I: C^3(\mathbb{R}) \rightarrow \mathbb{R}^3$, becomes invertible when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$.

Besides, the functions xe^x , e^x , and 1 are linearly independent (use Wronskian). It follows that $\ker(L) = \text{Span}(xe^x, e^x, 1)$.

General linear equations

Definition. A **linear equation** is an equation of the form

$$L(\mathbf{x}) = \mathbf{b},$$

where $L : V \rightarrow W$ is a linear mapping, \mathbf{b} is a given vector from W , and \mathbf{x} is an unknown vector from V .

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The kernel of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k,$$

where \mathbf{x}_0 is a particular solution, $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a basis for the kernel of L , and t_1, \dots, t_k are arbitrary scalars.

Example.
$$\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$$

$$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -1 \end{array} \right)$$

$$\begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \rightarrow C(\mathbb{R})$,

$$Lu = u''' - 2u'' + u'.$$

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

We already know that functions xe^x , e^x and 1 form a basis for the kernel of L . It remains to find a particular solution.

$$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}.$$

Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3.$$