

MATH 304

Linear Algebra

Lecture 37:

Orthogonal matrices (continued).

Rigid motions.

Rotations in space.

Orthogonal matrices

Definition. A square matrix A is called **orthogonal** if $AA^T = A^T A = I$, i.e., $A^T = A^{-1}$.

Theorem 1 If A is an $n \times n$ orthogonal matrix, then

- (i) columns of A form an orthonormal basis for \mathbb{R}^n ;
- (ii) rows of A also form an orthonormal basis for \mathbb{R}^n .

Idea of the proof: Entries of matrix $A^T A$ are dot products of columns of A . Entries of AA^T are dot products of rows of A .

Theorem 2 If A is an $n \times n$ orthogonal matrix, then

- (i) A is diagonalizable in the complexified vector space \mathbb{C}^n ;
- (ii) all eigenvalues λ of A satisfy $|\lambda| = 1$.

Example. $A_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- Eigenvalues: $\lambda_1 = \cos \phi + i \sin \phi = e^{i\phi}$,
 $\lambda_2 = \cos \phi - i \sin \phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$,
 $\mathbf{v}_2 = (1, i)$.
- $\lambda_2 = \overline{\lambda_1}$ and $\mathbf{v}_2 = \overline{\mathbf{v}_1}$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for \mathbb{C}^2 .

Consider a linear operator $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:

- (i) $\|L(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$;
- (ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
- (iii) the transformation L preserves distance between points:
 $\|L(\mathbf{x}) - L(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
- (iv) L preserves length of vectors and angle between vectors;
- (v) the matrix A is orthogonal;
- (vi) the matrix of L relative to any orthonormal basis is orthogonal;
- (vii) L maps some orthonormal basis for \mathbb{R}^n to another orthonormal basis;
- (viii) L maps any orthonormal basis for \mathbb{R}^n to another orthonormal basis.

Rigid motions

Definition. A transformation $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an **isometry** (or a **rigid motion**) if it preserves distances between points: $\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$.

Examples. • Translation: $f(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where \mathbf{x}_0 is a fixed vector.

• Isometric linear operator: $f(\mathbf{x}) = A\mathbf{x}$, where A is an orthogonal matrix.

• If f_1 and f_2 are two isometries, then the composition $f_2 \circ f_1$ is also an isometry.

Theorem Any isometry $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Suppose $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear isometric operator.

Theorem There exists an orthonormal basis for \mathbb{R}^n such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1 , and

$$R_j = \begin{pmatrix} \cos \phi_j & -\sin \phi_j \\ \sin \phi_j & \cos \phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$

Classification of linear isometries in \mathbb{R}^2 :

$$\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

rotation
about the origin

reflection
in a line

Determinant:

1

-1

Eigenvalues:

$e^{i\phi}$ and $e^{-i\phi}$

-1 and 1

Classification of linear isometries in \mathbb{R}^3 :

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}.$$

A = rotation about a line; B = reflection in a plane; C = rotation about a line combined with reflection in the orthogonal plane.

$$\det A = 1, \quad \det B = \det C = -1.$$

A has eigenvalues $1, e^{i\phi}, e^{-i\phi}$. B has eigenvalues $-1, 1, 1$. C has eigenvalues $-1, e^{i\phi}, e^{-i\phi}$.

Example. Consider a linear operator $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that acts on the standard basis as follows: $L(\mathbf{e}_1) = \mathbf{e}_2$, $L(\mathbf{e}_2) = \mathbf{e}_3$, $L(\mathbf{e}_3) = -\mathbf{e}_1$.

L maps the standard basis to another orthonormal basis, which implies that L is a rigid motion. The matrix of L

relative to the standard basis is $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

It is orthogonal, which is another proof that L is isometric.

It follows from the classification that the operator L is either a rotation about an axis, or a reflection in a plane, or the composition of a rotation about an axis with the reflection in the plane orthogonal to the axis.

$\det A = -1 < 0$ so that L reverses orientation. Therefore L is not a rotation. Further, $A^2 \neq I$ so that L^2 is not the identity map. Therefore L is not a reflection.

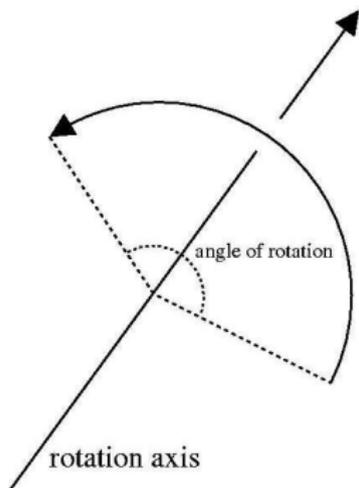
Hence L is a rotation about an axis composed with the reflection in the orthogonal plane. Then there exists an orthonormal basis for \mathbb{R}^3 such that the matrix of the operator L relative to that basis is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix},$$

where ϕ is the angle of rotation. Note that the latter matrix is similar to the matrix A . Similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). Therefore $\text{trace}(A) = -1 + 2 \cos \phi$. On the other hand, $\text{trace}(A) = 0$. Hence $-1 + 2 \cos \phi = 0$. Then $\cos \phi = 1/2$ so that $\phi = 60^\circ$.

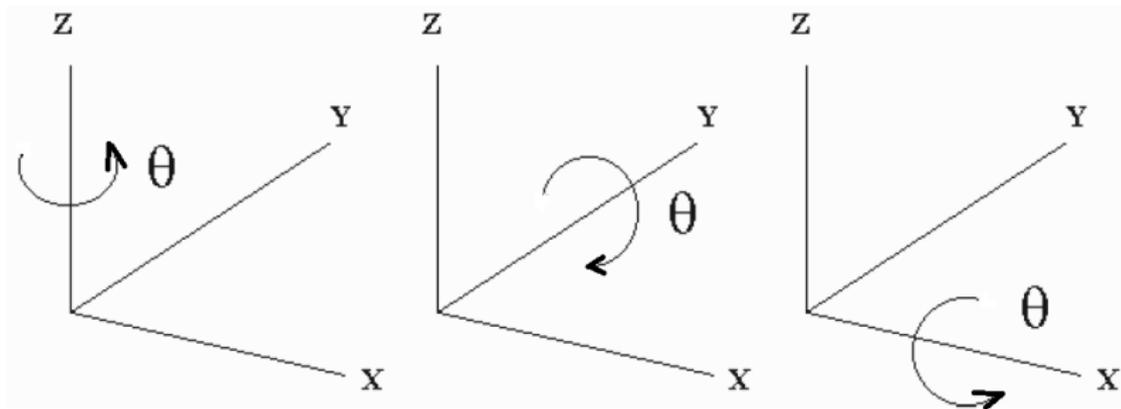
The axis of rotation consists of vectors \mathbf{v} such that $A\mathbf{v} = -\mathbf{v}$. In other words, this is the eigenspace of A associated to the eigenvalue -1 . One can find that the eigenspace is spanned by the vector $(1, -1, 1)$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Counterclockwise rotations about coordinate axes



$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

Problem. Find the matrix of the rotation by 90° about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of \mathbf{a} .

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is the matrix of (counterclockwise) rotation by 90° about the x -axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that \mathbf{v}_1 points in the same direction as \mathbf{a} . Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B will be the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (-2, -1, 2)$, and $\mathbf{c} = (2, -2, 1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

$$\det U = \frac{1}{27} \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case $\det U = -1$, we would change \mathbf{v}_3 to $-\mathbf{v}_3$, or change \mathbf{v}_2 to $-\mathbf{v}_2$, or interchange \mathbf{v}_2 and \mathbf{v}_3 .)

$$A = UBU^T$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.$$