MATH 304

Linear Algebra

Lecture 38:
Orthogonal polynomials.

Orthogonal polynomials

$\mathcal{P}$: the vector space of all polynomials with real coefficients: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.
Basis for $\mathcal{P}$: $1, x, x^2, \ldots, x^n, \ldots$

Suppose that $\mathcal{P}$ is endowed with an inner product.

Definition. **Orthogonal polynomials** (relative to the inner product) are polynomials $p_0, p_1, p_2, \ldots$ such that $\deg p_n = n$ (p_0 is a nonzero constant) and $\langle p_n, p_m \rangle = 0$ for $n \neq m$.

Remark. The orthogonal polynomials are linearly independent. It follows that $p_0, p_1, p_2, \ldots$ is a basis for $\mathcal{P}$.

Orthogonal polynomials can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2, \dots$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \dots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Then $p_0, p_1, p_2, \dots$ are orthogonal polynomials.

Theorem (a) Orthogonal polynomials always exist.

(b) The orthogonal polynomial of a fixed degree is unique up to scaling.

(c) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, q \rangle = 0$ for any polynomial q with $\deg q < \deg p$.

(d) A polynomial $p \neq 0$ is an orthogonal polynomial if and only if $\langle p, x^k \rangle = 0$ for any $0 \leq k < \deg p$.

Proof of statement (b): Suppose that P and R are two orthogonal polynomials of the same degree n. Then $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $R(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$, where $a_n, b_n \neq 0$. Consider a polynomial $Q(x) = b_n P(x) - a_n R(x)$. By construction, $\deg Q < n$. It follows from statement (c) that $\langle P, Q \rangle = \langle R, Q \rangle = 0$. Then

$$\langle Q, Q \rangle = \langle b_n P - a_n R, Q \rangle = b_n \langle P, Q \rangle - a_n \langle R, Q \rangle = 0,$$

which means that $Q = 0$. Thus $R(x) = (a_n^{-1} b_n)\, P(x)$.

Example. $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$

Note that $\langle x^m, x^n \rangle = \int_{-1}^{1} x^{m+n}\,dx = 0$ if $m + n$ is odd. Hence $p_{2k}(x)$ contains only even powers of x while $p_{2k+1}(x)$ contains only odd powers of x.

$$p_0(x) = 1,$$
$$p_1(x) = x,$$
$$p_2(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3},$$
$$p_3(x) = x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle}x = x^3 - \frac{3}{5}x.$$

$p_0, p_1, p_2, \ldots$ are called the **Legendre polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is $P_n(1) = 1$. In particular, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Problem. Find $P_4(x)$.

Let $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.
We know that $P_4(1) = 1$ and $\langle P_4, x^k \rangle = 0$ for $0 \leq k \leq 3$.

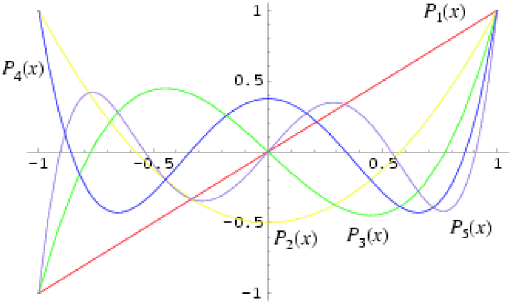
$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0$,
$\langle P_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0$, $\langle P_4, x \rangle = \frac{2}{5}a_3 + \frac{2}{3}a_1$,
$\langle P_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0$, $\langle P_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1$.

$$\begin{cases} a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases}$$

$$\begin{cases} \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases} \implies a_1 = a_3 = 0$$

$$\begin{cases} a_4 + a_2 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \end{cases} \iff \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}$$

Thus $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.



Legendre polynomials

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $p(x)$ that minimizes the distance relative to the integral norm

$$\|f - p\| = \left(\int_{-1}^{1} |f(x) - p(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of polynomials of degree at most 2.

The Legendre polynomials P_0, P_1, P_2 form an orthogonal basis for $\mathcal{P}_3$. Therefore

$$p(x) = \frac{\langle f, P_0\rangle}{\langle P_0, P_0\rangle} P_0(x) + \frac{\langle f, P_1\rangle}{\langle P_1, P_1\rangle} P_1(x) + \frac{\langle f, P_2\rangle}{\langle P_2, P_2\rangle} P_2(x).$$

$$\langle f, P_0\rangle = \int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 1,$$

$$\langle f, P_1\rangle = \int_{-1}^{1} |x|\, x\, dx = 0,$$

$$\langle f, P_2\rangle = \int_{-1}^{1} |x|\, \frac{3x^2-1}{2}\, dx = \int_0^1 x(3x^2-1)\, dx = \frac{1}{4},$$

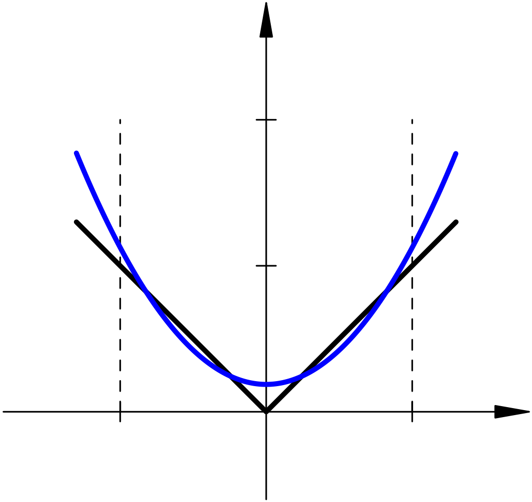
$$\langle P_0, P_0\rangle = \int_{-1}^{1} dx = 2, \qquad \langle P_2, P_2\rangle = \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 dx = \frac{2}{5}.$$

In general, $\langle P_n, P_n\rangle = \dfrac{2}{2n+1}$.

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

Solution: $p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x)$

$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$



How to evaluate orthogonal polynomials

Suppose $p_0, p_1, p_2, \dots$ are orthogonal polynomials with respect to an inner product of the form

$$\langle p, q \rangle = \int_a^b p(x)q(x)w(x)\,dx.$$

Theorem The polynomials satisfy recurrences

$$p_n(x) = (\alpha_n x + \beta_n)\,p_{n-1}(x) + \gamma_n\,p_{n-2}(x)$$

for all $n \geq 2$, where $\alpha_n, \beta_n, \gamma_n$ are some constants.

Recurrent formulas for the Legendre polynomials:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

For example, $4P_4(x) = 7xP_3(x) - 3P_2(x)$.

Definition. **Chebyshev polynomials** $T_0, T_1, T_2, \ldots$ are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1 - x^2}}\, dx,$$

with the standardization $T_n(1) = 1$.

Remark. “T” is like in “Tschebyscheff”.

Change of variable in the integral: $x = \cos\phi$.

$$\langle p, q \rangle = -\int_{0}^{\pi} \frac{p(\cos\phi)\, q(\cos\phi)}{\sqrt{1 - \cos^2\phi}} \cos'\phi\, d\phi$$

$$= \int_{0}^{\pi} p(\cos\phi)\, q(\cos\phi)\, d\phi.$$

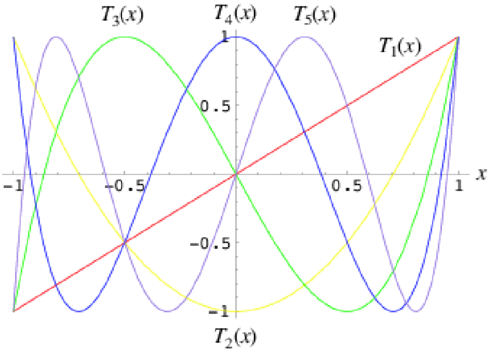
Theorem. $T_n(\cos\phi) = \cos n\phi$.

$$\langle T_n, T_m \rangle = \int_0^\pi T_n(\cos\phi)\, T_m(\cos\phi)\, d\phi$$
$$= \int_0^\pi \cos(n\phi)\cos(m\phi)\, d\phi = 0 \quad \text{if} \quad n \neq m.$$

Recurrent formula: $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

$T_0(x) = 1, \quad T_1(x) = x,$
$T_2(x) = 2x^2 - 1,$
$T_3(x) = 4x^3 - 3x,$
$T_4(x) = 8x^4 - 8x^2 + 1, \ \dots$

That is, $\cos 2\phi = 2\cos^2\phi - 1$,
$\cos 3\phi = 4\cos^3\phi - 3\cos\phi$,
$\cos 4\phi = 8\cos^4\phi - 8\cos^2\phi + 1, \ \dots$



Chebyshev polynomials