Quiz 2: Solution

Problem. Let L denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $L(\mathbf{e}_1) = \mathbf{e}_3$, $L(\mathbf{e}_2) = \mathbf{e}_1$, $L(\mathbf{e}_3) = \mathbf{e}_2$.

(i) Explain why L is a rigid motion.

(ii) Is L a rotation about an axis? Is L a reflection in a plane? Explain your answers.

(iii) If L is a rotation, find the axis and the angle. If L is a reflection, find the plane. If L is neither rotation nor reflection, describe the action of L in geometric terms.

The matrix of the operator L (relative to the standard basis) is

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This matrix is orthogonal since its columns form an orthonormal set (or, equivalently, since $M^T M = I$). Therefore L is a rigid motion. According to the classification of linear isometries in $\mathbb{R}^3$, L is either a rotation about an axis, or a reflection in a plane, or the composition of two. Since $\det M = 1 > 0$, the transformation L preserves orientation. Hence L is a rotation.

As L is a rotation about an axis, the matrix M is similar to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

where ϕ is the angle of rotation. Similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). The trace of M is 0. Hence $1 + 2\cos\phi = 0$. Then $\cos\phi = -1/2$ so that $\phi = 2\pi/3$.

The axis of the rotation L is the set of all points fixed by L. Since $L(\mathbf{v}) = M\mathbf{v}$ for all column vectors $\mathbf{v} \in \mathbb{R}^3$, the axis coincides with the eigenspace of the matrix M associated to the eigenvalue 1. To find the eigenspace, we convert the matrix $M - I$ into reduced row echelon form:

$$M - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now a vector $\mathbf{v} = (x, y, z)$ belongs to the eigenspace if and only if $x - z = y - z = 0$. The general solution of the system is $x = y = z = t$, where $t \in \mathbb{R}$. Thus the axis of rotation is the line spanned by the vector $(1, 1, 1)$.

Alternative solution: The operator L maps one orthonormal basis to an orthonormal basis (namely, the standard basis is mapped to itself). Therefore L is a rigid motion. According to the classification of linear isometries in $\mathbb{R}^3$, L is either a rotation about an axis, or a reflection in a plane, or the composition of two.

Note that $L^3(\mathbf{e}_1) = L(L(L(\mathbf{e}_1))) = L(L(\mathbf{e}_3)) = L(\mathbf{e}_2) = \mathbf{e}_1$. Likewise, $L^3(\mathbf{e}_2) = \mathbf{e}_2$ and $L^3(\mathbf{e}_3) = \mathbf{e}_3$. Since L^3 is linear, it is the identity map. Now it follows that L preserves orientation and so is a rotation. Let ϕ be the angle of rotation, $0 \leq \phi \leq \pi$. Then L^3 is a rotation by 3ϕ. Since L^3 is the identity, we obtain that $3\phi = 2\pi$.

The axis of the rotation L is the set of all points fixed by L. For any vector $(x, y, z) \in \mathbb{R}^3$ we have

$$L(x, y, z) = L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3) = x\mathbf{e}_3 + y\mathbf{e}_1 + z\mathbf{e}_2 = (y, z, x).$$

It follows that $L(x, y, z) = (x, y, z)$ if and only if $x = y = z$. Thus the axis of the rotation is the line spanned by the vector $(1, 1, 1)$.