

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) The planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$ intersect in a line. Find a parametric representation for the line.

To find the intersection set, we need to solve the system

$$\begin{cases} x + 2y + 2z = 1, \\ 4x + 7y + 4z = 5. \end{cases}$$

Let us convert the system to reduced form using elementary operations:

$$\begin{cases} x + 2y + 2z = 1 \\ 4x + 7y + 4z = 5 \end{cases} \iff \begin{cases} x + 2y + 2z = 1 \\ -y - 4z = 1 \end{cases} \iff \begin{cases} x + 2y + 2z = 1 \\ y + 4z = -1 \end{cases} \iff \begin{cases} x - 6z = 3 \\ y + 4z = -1 \end{cases}$$

It follows that the general solution of the system is $x = 6t + 3$, $y = -4t - 1$, $z = t$, where $t \in \mathbb{R}$. Therefore the two planes intersect in the line $t(6, -4, 1) + (3, -1, 0)$.

Problem 2 (20 pts.) Consider a linear operator $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator L .

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v} \cdot \mathbf{v}_1 = x + y + z$ and $L(\mathbf{v}) = (x + y + z, 2(x + y + z), 2(x + y + z))$. Let A denote the matrix of the linear operator L . The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for \mathbb{R}^3 . Therefore

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

(ii) Find the dimensions of the image and the null-space of L .

The image $\text{Im } L$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is easy to see that $\text{Im } L$ is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$. Hence $\dim \text{Im } L = 1$.

The null-space $\text{Null } L$ of the operator L is the subspace of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Clearly, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{v}_1 = 0$. Therefore $\text{Null } L$ is the plane $x + y + z = 0$ orthogonal to \mathbf{v}_1 and passing through the origin. Its dimension is 2.

(iii) Find bases for the image and the null-space of L .

Since the image of L is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$, this vector is a basis for the image. The null-space of L is the plane given by the equation $x + y + z = 0$. The general solution of the equation is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. It gives rise to a parametric representation $t(-1, 1, 0) + s(-1, 0, 1)$ of the plane. Thus the null-space of L is spanned by the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$. Since the two vectors are linearly independent, they form a basis for $\text{Null } L$.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A .

The determinant can be evaluated using row or column expansions. For example, let us expand the determinant of A by the first row:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{vmatrix}.$$

Then expand each of the two 3-by-3 determinants by the third row:

$$\det A = \left(\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \right) - \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (0 + (-3)) - (-2) = -1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires more work but we are going to do it anyway, to find the inverse of A .

(ii) Find the inverse matrix A^{-1} .

First we merge the matrix A with the identity matrix into one 4-by-8 matrix:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Interchange the third row with the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract the second row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 1 \end{array} \right).$$

Subtract 2 times the third row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right).$$

Add the fourth row to the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right).$$

Subtract 2 times the fourth row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -4 & 4 & 3 & -2 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right).$$

Add 2 times the third row to the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -4 & 4 & 3 & -2 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right).$$

Subtract the second row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array} \right).$$

Finally the left part of our 4-by-8 matrix is transformed into the identity matrix. Therefore the current right side is the inverse matrix of A . Thus

$$A^{-1} = \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{array} \right)^{-1} = \left(\begin{array}{cccc} 3 & -2 & -1 & 0 \\ -2 & 2 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & -2 & -1 & 1 \end{array} \right).$$

As a byproduct, we can evaluate the determinant of A . We have transformed A into the identity matrix using elementary row operations. These included one row exchange and no row multiplications. It follows that $\det A = -\det I = -1$.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B .

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. One obtains that

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 - 3(1 - \lambda) + 2 \\ &= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1 - \lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3 - \lambda). \end{aligned}$$

Hence the matrix B has two eigenvalues: 0 and 3.

(ii) Find a basis for \mathbb{R}^3 consisting of eigenvectors of B ?

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$. Hence the eigenspace of B associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$.

Now consider the case $\lambda = 3$. We obtain that

$$\begin{aligned} (B - 3I)\mathbf{x} = \mathbf{0} &\iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\iff \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0, \\ y - z = 0. \end{cases} \end{aligned}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B . They are linearly independent since the matrix whose rows are these vectors is invertible:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for \mathbb{R}^3 .

(iii) Find an orthonormal basis for \mathbb{R}^3 consisting of eigenvectors of B ?

It is easy to check that the vector \mathbf{v}_3 is orthogonal to \mathbf{v}_1 and \mathbf{v}_2 . To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Namely, we replace the vector \mathbf{v}_2 by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for \mathbb{R}^3 . Since \mathbf{u} is a linear combination of the vectors \mathbf{v}_1 and \mathbf{v}_2 , it is also an eigenvector of B associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\mathbf{w}_2 = \frac{\mathbf{u}}{|\mathbf{u}|}$, and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|}$ form an orthonormal basis for \mathbb{R}^3 consisting of eigenvectors of B . We get that $|\mathbf{v}_1| = \sqrt{2}$, $|\mathbf{u}| = \sqrt{3/2}$, and $|\mathbf{v}_3| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1, -1, 2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

Problem 5 (20 pts.) Let V be a three-dimensional subspace of \mathbb{R}^4 spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (1, 3, 1, 1)$, and $\mathbf{x}_3 = (1, 1, -3, -1)$.

(i) Find an orthogonal basis for V .

To find an orthogonal basis for the subspace V , we apply the Gram-Schmidt orthogonalization process to the spanning set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 0, 0),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (1, 3, 1, 1) - \frac{4}{2}(1, 1, 0, 0) = (-1, 1, 1, 1),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = (1, 1, -3, -1) - \frac{2}{2}(1, 1, 0, 0) - \frac{-4}{4}(-1, 1, 1, 1) = (-1, 1, -2, 0).$$

By construction, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthogonal to each other. It follows that they are linearly independent. Also, the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the same as the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, that is, the subspace V . Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V .

(ii) Find the distance from the point $\mathbf{y} = (2, 0, 2, 4)$ to the subspace V .

The distance from \mathbf{y} to V is the distance from \mathbf{y} to the closest point \mathbf{y}_0 in V , which is the orthogonal projection of \mathbf{y} on V . Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V , it follows that

$$\begin{aligned} \mathbf{y}_0 &= \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 \\ &= \frac{2}{2}(1, 1, 0, 0) + \frac{4}{4}(-1, 1, 1, 1) + \frac{-6}{6}(-1, 1, -2, 0) = (1, 1, 3, 1). \end{aligned}$$

Then $\mathbf{y} - \mathbf{y}_0 = (2, 0, 2, 4) - (1, 1, 3, 1) = (1, -1, -1, 3)$ and the desired distance is $|\mathbf{y} - \mathbf{y}_0| = \sqrt{12} = 2\sqrt{3}$.

Bonus Problem 6 (15 pts.) Let S be the set of all points in \mathbb{R}^3 that lie at same distance from the planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$. Show that S is the union of two planes and find these planes.

For any point $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ let $d_1(\mathbf{x}_0)$ denote the distance from \mathbf{x}_0 to the plane $x + 2y + 2z = 1$ and $d_2(\mathbf{x}_0)$ denote the distance from \mathbf{x}_0 to the plane $4x + 7y + 4z = 5$. Then

$$\begin{aligned} d_1(\mathbf{x}_0) &= \frac{|x_0 + 2y_0 + 2z_0 - 1|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3} |x_0 + 2y_0 + 2z_0 - 1|, \\ d_2(\mathbf{x}_0) &= \frac{|4x_0 + 7y_0 + 4z_0 - 5|}{\sqrt{4^2 + 7^2 + 4^2}} = \frac{1}{9} |4x_0 + 7y_0 + 4z_0 - 5|. \end{aligned}$$

The point \mathbf{x}_0 belongs to the set S if $d_1(\mathbf{x}_0) = d_2(\mathbf{x}_0)$, i.e., if

$$\frac{1}{3}|x_0 + 2y_0 + 2z_0 - 1| = \frac{1}{9}|4x_0 + 7y_0 + 4z_0 - 5|.$$

This means that

$$3(x_0 + 2y_0 + 2z_0 - 1) = 4x_0 + 7y_0 + 4z_0 - 5$$

or

$$3(x_0 + 2y_0 + 2z_0 - 1) = -(4x_0 + 7y_0 + 4z_0 - 5).$$

Equivalently, $x_0 + y_0 - 2z_0 = 2$ or $7x_0 + 13y_0 + 10z_0 = 8$.

Thus the set S is the union of the planes $x + y - 2z = 2$ and $7x + 13y + 10z = 8$.

Bonus Problem 7 (20 pts.) Find a quadratic polynomial that is an orthogonal polynomial relative to the inner product

$$\langle p, q \rangle = \int_0^1 xp(x)q(x) dx.$$

First observe that for any integers $m, n \geq 0$,

$$\langle x^m, x^n \rangle = \int_0^1 x^{m+n+1} dx = \frac{1}{m+n+2}.$$

To get the first three orthogonal polynomials, we apply the Gram-Schmidt orthogonalization process to the polynomials 1, x , and x^2 :

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x - \frac{1/3}{1/2} = x - \frac{2}{3}, \\ p_2(x) &= x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} \left(x - \frac{2}{3}\right). \end{aligned}$$

Evaluating inner products in the latter formula, we obtain that

$$\begin{aligned} \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} &= \frac{1/4}{1/2} = \frac{1}{2}, \\ \langle x^2, p_1 \rangle &= \langle x^2, x \rangle - \frac{2}{3} \langle x^2, 1 \rangle = \frac{1}{5} - \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}, \\ \langle p_1, p_1 \rangle &= \langle x, x \rangle - 2 \cdot \frac{2}{3} \langle x, 1 \rangle + \left(\frac{2}{3}\right)^2 \langle 1, 1 \rangle = \frac{1}{4} - 2 \cdot \frac{2}{3} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{2} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}, \\ \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} &= \frac{1/30}{1/36} = \frac{6}{5}. \end{aligned}$$

Thus

$$p_2(x) = x^2 - \frac{1}{2} - \frac{6}{5} \left(x - \frac{2}{3}\right) = x^2 - \frac{6}{5}x - \frac{1}{2} + \frac{4}{5} = x^2 - \frac{6}{5}x + \frac{3}{10}$$

is the desired orthogonal polynomial.

Alternative solution: A quadratic polynomial $p(x) = x^2 + ax + b$ is an orthogonal polynomial if $\langle p, q \rangle = 0$ for any polynomial q such that $\deg q < \deg p$. Actually, it is enough to require that $\langle p, 1 \rangle = \langle p, x \rangle = 0$. Note that

$$\langle p, 1 \rangle = \int_0^1 (x^3 + ax^2 + bx) dx = \frac{1}{4} + \frac{a}{3} + \frac{b}{2},$$

$$\langle p, x \rangle = \int_0^1 (x^4 + ax^3 + bx^2) dx = \frac{1}{5} + \frac{a}{4} + \frac{b}{3}.$$

Hence $p(x)$ is an orthogonal polynomial if and only if the coefficients a and b satisfy the following system:

$$\begin{cases} a/3 + b/2 = -1/4, \\ a/4 + b/3 = -1/5. \end{cases}$$

Solving the system, we obtain

$$\begin{aligned} \begin{cases} a/3 + b/2 = -1/4 \\ a/4 + b/3 = -1/5 \end{cases} &\iff \begin{cases} 2a + 3b = -1.5 \\ 3a + 4b = -2.4 \end{cases} \iff \begin{cases} 2a + 3b = -1.5 \\ a + b = -0.9 \end{cases} \\ &\iff \begin{cases} b = 0.3 \\ a + b = -0.9 \end{cases} \iff \begin{cases} a = -1.2 \\ b = 0.3 \end{cases} \end{aligned}$$

Thus $p(x) = x^2 - 1.2x + 0.3$ is an orthogonal polynomial.