MATH 311-504

Topics in Applied Mathematics

Lecture 13: Review for Test 1.

Topics for Test 1

Vectors (Williamson/Trotter 1.1–1.2, 1.4, 1.6, 2.2C)

- Vector addition and scalar multiplication
- Length of a vector, angle between vectors
- Dot product, orthogonality
- Cross product, mixed triple product
- Linear dependence

Analytic geometry (Williamson/Trotter 1.3, 1.5–1.6)

- Lines and planes, parametric representation
- Equations of a line in $\mathbb{R}^2$ and of a plane in $\mathbb{R}^3$
- Distance from a point to a line in $\mathbb{R}^2$ or from a point to a plane in $\mathbb{R}^3$
- Area of a triangle and a parallelogram in $\mathbb{R}^3$
- Volume of a parallelepiped in $\mathbb{R}^3$

Topics for Test 1

Systems of linear equations (Williamson/Trotter 2.1–2.2)

- Elimination and back substitution
- Elementary operations, Gaussian elimination
- Matrix of coefficients and augmented matrix
- Elementary row operations
- Row echelon form and reduced row echelon form
- Free variables, parametric representation of the solution set
- Homogeneous systems, checking for linear independence of vectors

Topics for Test 1

Matrix algebra (Williamson/Trotter 2.3–2.4)

- Matrix addition and scalar multiplication
- Matrix multiplication
- Diagonal matrices, identity matrix
- Matrix polynomials
- Inverse matrix

Determinants (Williamson/Trotter 2.5)

- Explicit formulas for 2×2 and 3×3 matrices
- Elementary row and column operations
- Row and column expansions
- Test for linear dependence

Sample problems for Test 1

Problem 1 (25 pts.) Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2,0,0)$, $(1,1,0)$, and $(-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(i) Find a parametric representation for the line ℓ.
(ii) Find a parametric representation for the plane Π.
(iii) Find an equation for the plane Π.
(iv) Find the point where the line ℓ intersects the plane Π.
(v) Find the angle between the line ℓ and the plane Π.
(vi) Find the distance from the origin to the plane Π.

Problem 2 (15 pts.) Let $f(x) = a\cos 2x + b\cos x + c$. Find a, b, and c so that $f(0) = 0$, $f''(0) = 2$, and $f''''(0) = 10$.

Sample problems for Test 1

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Find the inverse matrix A^{-1}.

Problem 4 (20 pts.) Evaluate the following determinants:

$$\text{(i)} \begin{vmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}, \qquad \text{(ii)} \begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix}.$$

Bonus Problem 5 (15 pts.) Find the volume of the tetrahedron with vertices at the points $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,1,0)$, $\mathbf{c} = (0,0,1)$, and $\mathbf{d} = (2,3,5)$.

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2, 0, 0)$, $(1, 1, 0)$, and $(-3, 0, 2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1, 1, 1)$ in the direction $(2, 2, 2)$.

(i) Find a parametric representation for the line ℓ.

Parametric representation: $t(2, 2, 2) + (1, 1, 1)$.

The line ℓ passes through the origin ($t = -1/2$).

Hence an equivalent representation is $s(2, 2, 2)$.

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2,0,0)$, $(1,1,0)$, and $(-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(ii) Find a parametric representation for the plane Π.

Since the plane Π contains the points $\mathbf{a} = (2,0,0)$, $\mathbf{b} = (1,1,0)$, and $\mathbf{c} = (-3,0,2)$, the vectors $\mathbf{b} - \mathbf{a} = (-1,1,0)$ and $\mathbf{c} - \mathbf{a} = (-5,0,2)$ are parallel to Π. Clearly, $\mathbf{b} - \mathbf{a}$ is not parallel to $\mathbf{c} - \mathbf{a}$. Hence we get a parametric representation

$$t_1(\mathbf{b} - \mathbf{a}) + t_2(\mathbf{c} - \mathbf{a}) + \mathbf{a} =$$
$$= t_1(-1,1,0) + t_2(-5,0,2) + (2,0,0).$$

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $\mathbf{a} = (2,0,0)$, $\mathbf{b} = (1,1,0)$, and $\mathbf{c} = (-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(iii) Find an equation for the plane Π.

Vectors $\mathbf{b}-\mathbf{a} = (-1,1,0)$ and $\mathbf{c}-\mathbf{a} = (-5,0,2)$ are parallel to Π $\implies$ their cross product $\mathbf{p}$ is orthogonal to Π.

$$\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -5 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 0 \\ -5 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 1 \\ -5 & 0 \end{vmatrix} \mathbf{k}$$

$$= 2\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = (2,2,5).$$

A point $\mathbf{x} = (x,y,z)$ is in the plane Π if and only if

$\mathbf{p}\cdot(\mathbf{x}-\mathbf{a}) = 0 \iff 2(x-2) + 2y + 5z = 0$

$\iff 2x + 2y + 5z = 4$

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2, 0, 0)$, $(1, 1, 0)$, and $(-3, 0, 2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1, 1, 1)$ in the direction $(2, 2, 2)$.

(iv) Find the point where the line ℓ intersects the plane Π.

Let $\mathbf{x}_0 = (x, y, z)$ be the point of intersection. Then $\mathbf{x}_0 = s(2, 2, 2)$ for some $s \in \mathbb{R}$ and also $2x + 2y + 5z = 4$.

$$2(2s) + 2(2s) + 5(2s) = 4 \iff s = 2/9$$

Hence $\mathbf{x}_0 = (4/9, 4/9, 4/9)$.

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2,0,0)$, $(1,1,0)$, and $(-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(v) Find the angle between the line ℓ and the plane Π.

Let ϕ denote the angle between vectors $\mathbf{u} = (2,2,2)$ and $\mathbf{p} = (2,2,5)$. Our angle is $\psi = |\pi/2 - \phi|$.

$$\cos\phi = \frac{\mathbf{u}\cdot\mathbf{p}}{|\mathbf{u}|\,|\mathbf{p}|} = \frac{18}{\sqrt{12}\,\sqrt{33}} = \frac{3}{\sqrt{11}}$$

$$\psi = \frac{\pi}{2} - \arccos\frac{3}{\sqrt{11}} = \arcsin\frac{3}{\sqrt{11}}$$

Problem 1 Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2, 0, 0)$, $(1, 1, 0)$, and $(-3, 0, 2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1, 1, 1)$ in the direction $(2, 2, 2)$.

(vi) Find the distance from the origin to the plane Π.

The equation of the plane Π is $2x + 2y + 5z = 4$. Hence the distance from a point (x_0, y_0, z_0) to Π equals

$$\frac{|2x_0 + 2y_0 + 5z_0 - 4|}{\sqrt{2^2 + 2^2 + 5^2}} = \frac{|2x_0 + 2y_0 + 5z_0 - 4|}{\sqrt{33}}.$$

The distance from the origin to the plane is equal to $4/\sqrt{33}$.

Bonus Problem 5. Find the volume of the tetrahedron with vertices $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (0, 0, 1)$, and $\mathbf{d} = (2, 3, 5)$.

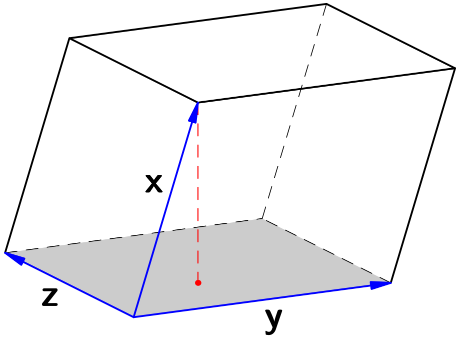
Vectors $\mathbf{x} = \mathbf{b} - \mathbf{a} = (-1, 1, 0)$, $\mathbf{y} = \mathbf{c} - \mathbf{a} = (-1, 0, 1)$, and $\mathbf{z} = \mathbf{d} - \mathbf{a} = (1, 3, 5)$ are represented by adjacent edges of the tetrahedron.

It follows that the volume of the tetrahedron is $\frac{1}{6}\big|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})\big|$.

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 5 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 1 \\ 3 & 5 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 1 & 5 \end{vmatrix} = 9.$$

Thus the volume of the tetrahedron is

$$\frac{1}{6}\big|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})\big| = \frac{1}{6} \cdot 9 = 1.5.$$

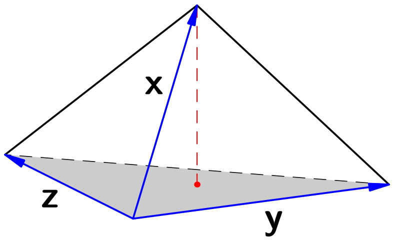


Parallelepiped is a prism.

(Volume) $=$ (area of the base) $\times$ (height)

Area of the base $= |\mathbf{y} \times \mathbf{z}|$

Volume $= |\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$



Tetrahedron is a pyramid.

$$(\text{Volume}) = \frac{1}{3}(\text{area of the base}) \times (\text{height})$$

$$\text{Area of the base} = \frac{1}{2}|\mathbf{y} \times \mathbf{z}|$$

$$\implies \quad \text{Volume} = \frac{1}{6}|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$$

Problem 2. Let $f(x) = a\cos 2x + b\cos x + c$. Find a, b, and c so that $f(0) = 0$, $f''(0) = 2$, and $f''''(0) = 10$.

$$f''(x) = -4a\cos 2x - b\cos x, \quad f''''(x) = 16a\cos 2x + b\cos x$$
$$\implies f(0) = a + b + c, \; f''(0) = -4a - b, \; f''''(0) = 16a + b.$$

The coefficients a, b, c should satisfy a system

$$\begin{cases} a + b + c = 0 \\ -4a - b = 2 \\ 16a + b = 10 \end{cases} \iff \begin{cases} a + b + c = 0 \\ -4a - b = 2 \\ 12a = 12 \end{cases} \iff \begin{cases} a = 1 \\ b = -6 \\ c = 5 \end{cases}$$

Thus $f(x) = \cos 2x - 6\cos x + 5$.

Problem 3. Let $A = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Find A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A\,|\,I) = \left(\begin{array}{cccc|cccc} 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Interchange the 1st row with the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Subtract the 1st row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Add the 4th row to the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

Add 2 times the 2nd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 16 & -1 & 3 & 2 & 0 & -4 \end{array}\right)$$

Add 16 times the 3rd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 3 & 2 & 16 & -20 \end{array}\right)$$

Multiply the 3rd and the 4th rows by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right)$$

Add the 4th row to the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right)$$

Subtract the 4th row from the 1st row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right)$$

Subtract 6 times the 3rd row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -3 & -2 & -16 & 20 \end{pmatrix}.$$

Problem 4(i). $A = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Find $\det A$.

In the solution of Problem 3, the matrix A has been transformed into the identity matrix using elementary row operations.

Those included one row exchange and two row multiplications, each time by -1.

$$\Longrightarrow \quad \det I = -(-1)^2 \det A$$

$$\Longrightarrow \quad \det A = -\det I = -1$$

Problem 4(ii). $B = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$. Find $\det B$.

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}.$$

Subtract 2 times the 2nd row from the 1st row:

Subtract 2 times the 2nd row from the 1st row:

$$\det B = -2\begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2\begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}.$$

Expand the determinant by the 1st row:

$$\det B = -2\begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2\cdot 9\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}.$$

Thus

$$\det B = -18\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18\cdot 2 = -36.$$