MATH 311-504

Topics in Applied Mathematics

# Lecture 1:
# Vectors. Dot product.

# Vectors

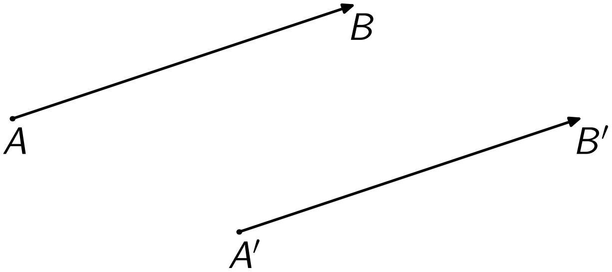
**Vector** is a mathematical concept characterized by its *magnitude* and *direction*.

**Scalar** is a mathematical concept characterized by its *magnitude* and, possibly, *sign*.
Scalar is a real number (positive or negative).

Many physical quantities are vectors:

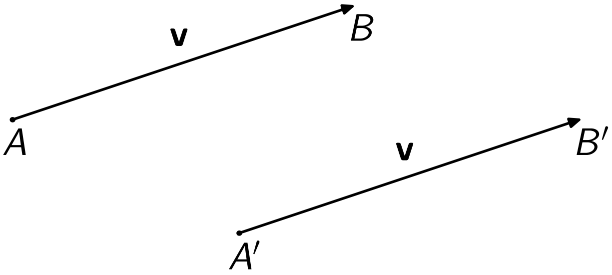
- force;
- displacement, velocity, acceleration;
- electric field, magnetic field.

# Vectors: geometric approach




- A vector is represented by a directed segment.
- Directed segment is drawn as an arrow.
- Different arrows represent the same vector if they are of the same length and direction.

# Vectors: geometric approach

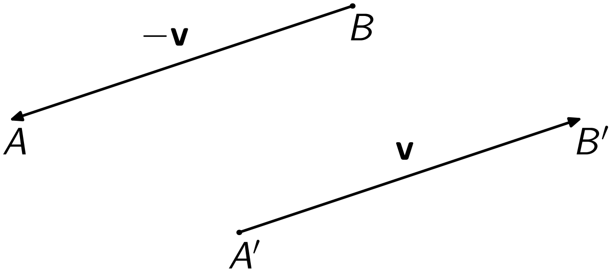



*Notation:* $\mathbf{v}$ or $\vec{v}$.

$\overrightarrow{AB}$ denotes the vector represented by the arrow with tip at $B$ and tail at $A$.

$\overrightarrow{AA}$ is called the *zero vector* and denoted $\mathbf{0}$ or $\vec{0}$.

# Vectors: geometric approach

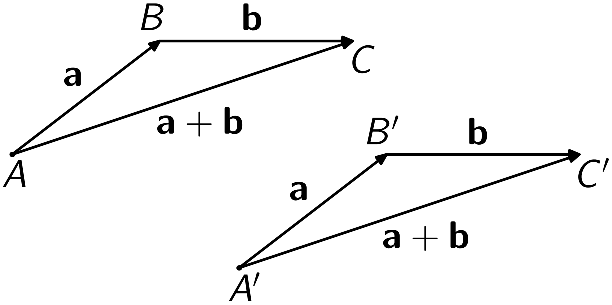



If $\mathbf{v} = \overrightarrow{AB}$ then $\overrightarrow{BA}$ is called the *inverse vector* of $\mathbf{v}$ and denoted $-\mathbf{v}$.

## Vector addition

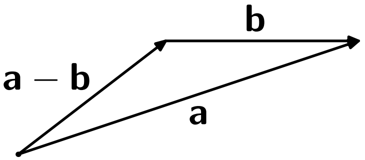
Given vectors $\mathbf{a}$ and $\mathbf{b}$, their *sum* $\mathbf{a}+\mathbf{b}$ is defined by the rule $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

That is, choose points $A, B, C$ so that $\overrightarrow{AB}=\mathbf{a}$ and $\overrightarrow{BC}=\mathbf{b}$. Then $\mathbf{a}+\mathbf{b}=\overrightarrow{AC}$.



# Vector subtraction

The *difference* of the two vectors is defined as
$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.



*Properties of vector addition:*

$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ (associative law)

$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ (commutative law)

$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$

$\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$

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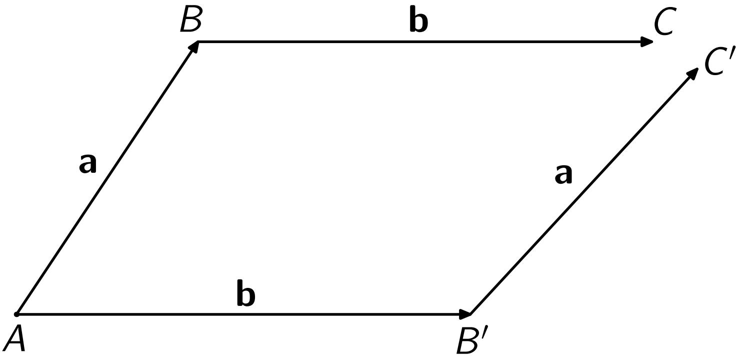
Let $\overrightarrow{AB}=\mathbf{a}$. Then $\mathbf{a}+\mathbf{0}=\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AB}=\mathbf{a}$,
$\mathbf{a}+(-\mathbf{a})=\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\mathbf{0}$.

Let $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$, and $\overrightarrow{CD}=\mathbf{c}$. Then
$(\mathbf{a}+\mathbf{b})+\mathbf{c}=(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$,
$\mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$.

# Parallelogram rule

Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{AB'} = \mathbf{b}$, and $\overrightarrow{B'C'} = \mathbf{a}$.
Then $\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$, $\mathbf{b} + \mathbf{a} = \overrightarrow{AC'}$.




*Wrong picture!*

# Parallelogram rule

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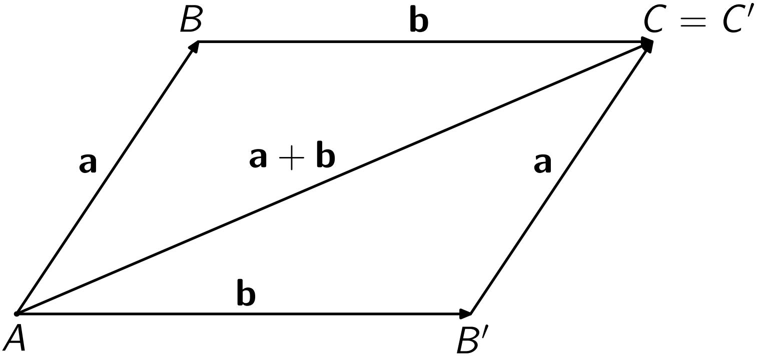
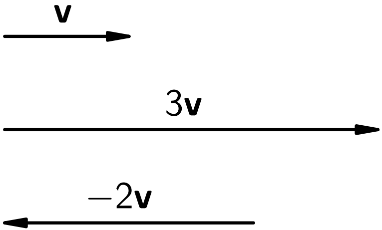



*Right picture!*

# Scalar multiplication

Let $\mathbf{v}$ be a vector and $r \in \mathbb{R}$. By definition, $r\mathbf{v}$ is a vector whose magnitude is $|r|$ times the magnitude of $\mathbf{v}$. The direction of $r\mathbf{v}$ coincides with that of $\mathbf{v}$ if $r > 0$. If $r < 0$ then the directions of $r\mathbf{v}$ and $\mathbf{v}$ are opposite.



*Properties of scalar multiplication:*

$r(s\mathbf{a}) = (rs)\mathbf{a}$ (associative law)

$r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$ (distributive law #1)

$(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$ (distributive law #2)

$1\mathbf{a} = \mathbf{a}$

$(-1)\mathbf{a} = -\mathbf{a}$

$0\mathbf{a} = \mathbf{0}$

# Length of a vector

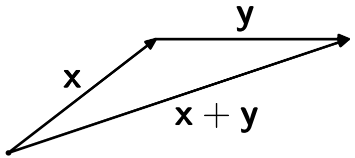
The **length** (or the **magnitude**) of a vector $\overrightarrow{AB}$ is the length of the representing segment $AB$. The length of a vector $\mathbf{v}$ is denoted $|\mathbf{v}|$.

*Properties of vector length:*

$|\mathbf{x}| \geq 0,\ \ |\mathbf{x}| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$|r\mathbf{x}| = |r|\,|\mathbf{x}|$ (homogeneity)

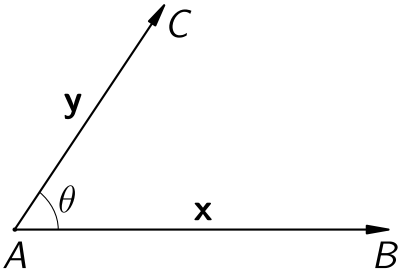
$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)

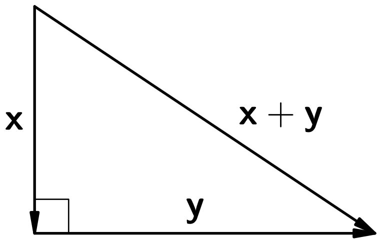


# Angle between vectors

Given nonzero vectors **x** and **y**, let $A$, $B$, and $C$ be points such that $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AC} = \mathbf{y}$. Then $\angle BAC$ is called the **angle** between **x** and **y**.

The vectors **x** and **y** are called **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if the angle between them equals $90^{\mathrm{o}}$.





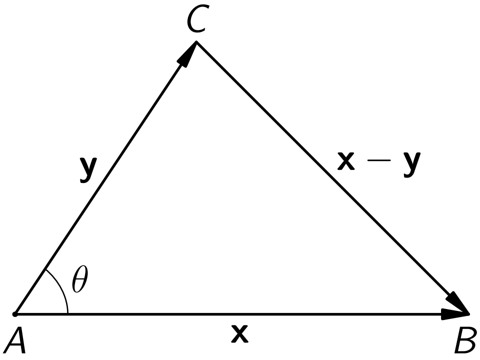

*Pythagorean Theorem:*

$$\mathbf{x} \perp \mathbf{y} \implies |\mathbf{x} + \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

3-*dimensional Pythagorean Theorem:*

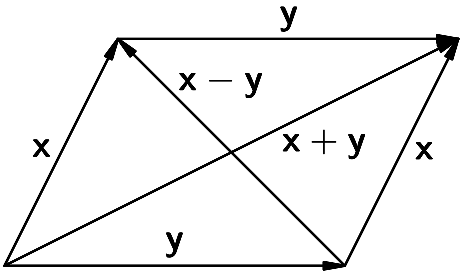
If vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are pairwise orthogonal then

$$|\mathbf{x} + \mathbf{y} + \mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2$$




*Law of cosines:*

$$|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}|\,|\mathbf{y}|\,\cos\theta$$




*Parallelogram Identity:*

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

# Dot product

The **dot product** of vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos\theta,$$

where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

The dot product is also called the **scalar product**.
Alternative notation: $(\mathbf{x}, \mathbf{y})$ or $\langle \mathbf{x}, \mathbf{y} \rangle$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

*Relations between lengths and dot products:*

- $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
- $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$
- $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x} \cdot \mathbf{y}$

# Vectors: algebraic approach

An $n$-dimensional vector is an element of $\mathbb{R}^n$, i.e., an ordered $n$-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. Components of the vector are called *coordinates*.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be vectors, and $r \in \mathbb{R}$ be a scalar. Then, by definition,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

$$r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n),$$

$$\mathbf{0} = (0, 0, \ldots, 0),$$

$$-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n),$$

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

*Properties of vector addition and scalar multiplication:*

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$$

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$$

$$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$$

$$\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$$

$$r(s\mathbf{a})=(rs)\mathbf{a}$$

$$r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b}$$

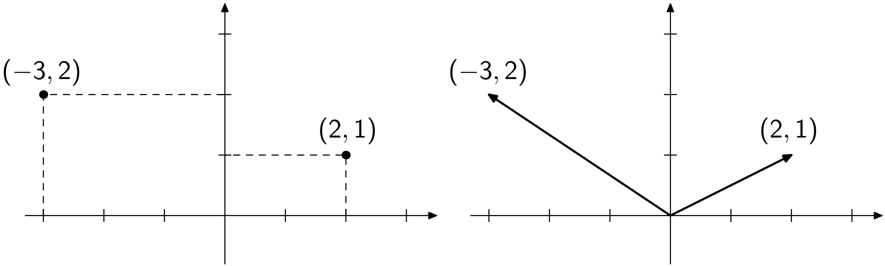
$$(r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a}$$

$$1\mathbf{a}=\mathbf{a}$$

$$(-1)\mathbf{a}=-\mathbf{a}$$

$$0\mathbf{a}=\mathbf{0}$$

# Cartesian coordinates: geometry meets algebra




Cartesian coordinates allow us to identify a line, a plane, and space with $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, respectively. Once we specify the *origin* $O$, each point $A$ is associated a *position vector* $\overrightarrow{OA}$. Conversely, every vector has a unique representative with tail at $O$.

## Standard basis

The *standard basis* in $\mathbb{R}^n$ is the set of $n$ vectors $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0, 0)$, ..., $\mathbf{e}_n = (0, 0, 0, \dots, 0, 1)$.

If $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

We say that $\mathbf{x}$ is a *linear combination* of vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. The numbers $x_1, x_2, \dots, x_n$ are called *coordinates* of $\mathbf{x}$. The vectors $x_1\mathbf{e}_1, x_2\mathbf{e}_2, \dots, x_n\mathbf{e}_n$ are called *components* of $\mathbf{x}$.

In $\mathbb{R}^2$, we have an alternative notation $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. In $\mathbb{R}^3$, we have an alternative notation $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

## Length and distance

*Definition.* The **length** of a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The **distance** between vectors (or points) $\mathbf{x}$ and $\mathbf{y}$ is $|\mathbf{y} - \mathbf{x}|$.

*Properties of length:*

$|\mathbf{x}| \geq 0$, $|\mathbf{x}| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$|r\mathbf{x}| = |r|\,|\mathbf{x}|$ (homogeneity)

$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)

# Dot product

*Definition.* The **dot product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k .$$

*Properties of dot product:*

$\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

*Relations between lengths and dot products:*

$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}|\,|\mathbf{y}|$ (Cauchy-Schwarz inequality)

$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|} \quad \text{for some} \quad 0 \leq \theta \leq \pi.$$

$\theta$ is called the **angle** between the vectors **x** and **y**. The vectors **x** and **y** are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).