MATH 311-504

Topics in Applied Mathematics

Lecture 2-12:
Bases of eigenvectors (continued).
Change of coordinates.

Diagonalization

Let $L : V \to V$ be a linear operator.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and A be the matrix of the operator L with respect to this basis.

Theorem The matrix A is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors of L.

If this is the case, then the diagonal entries of the matrix A are the corresponding eigenvalues of L.

$$L(\mathbf{v}_i) = \lambda_i \mathbf{v}_i \iff A = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

Eigenvalues and eigenvectors of a matrix

Eigenvalues λ of a square matrix A are roots of the characteristic equation $\det(A - \lambda I) = 0$.

Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Theorem Let A be an n-by-n matrix. Then $\det(A - \lambda I)$ is a polynomial of λ of degree n:

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n.$$

Corollary Any n-by-n matrix has at most n eigenvalues.

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of a linear operator L associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Corollary Suppose A is an n-by-n matrix that has n distinct eigenvalues. Then $\mathbb{R}^n$ has a basis consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.

- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1, 1)$.

- The eigenspace of A associated with the eigenvalue 3 is the line $t(1, 1)$.

- Eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$ of the matrix A form a basis for $\mathbb{R}^2$.

- Matrix of the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\mathbf{x}) = A\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ is $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

Example. $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 0 and 2.
- The eigenspace of A associated with the eigenvalue 0 is the line $t(-1,1,0)$.
- The eigenspace of A associated with the eigenvalue 2 is the plane $t(1,1,0) + s(-1,0,1)$.
- Eigenvectors $\mathbf{u}_1 = (-1,1,0)$, $\mathbf{u}_2 = (1,1,0)$, and $\mathbf{u}_3 = (-1,0,1)$ of the matrix A form a basis for $\mathbb{R}^3$.
- Matrix of the operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = A\mathbf{x}$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

There are **two obstructions** to diagonalization of a matrix (i.e., existence of a basis of eigenvectors). They are illustrated by the following examples.

Example 1. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\implies$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

Example 2. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Change of coordinates

Given a vector $\mathbf{v} \in \mathbb{R}^2$, let (x, y) be its standard coordinates, i.e., coordinates with respect to the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and let (x', y') be its coordinates with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Problem. Find a relation between (x, y) and (x', y').

By definition, $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{v}_1 + y'\mathbf{v}_2$.
In standard coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x' \begin{pmatrix} 3 \\ 1 \end{pmatrix} + y' \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\implies \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Change of coordinates

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccccc} & & V & & \\ & {\scriptstyle g_1}\swarrow & & \searrow{\scriptstyle g_2} & \\ \mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n \end{array}$$

The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself. It is represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Problem. Find the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbb{R}^3$ to the basis $\mathbf{u}_1 = (-1, 1, 0)$, $\mathbf{u}_2 = (1, 1, 0)$, $\mathbf{u}_3 = (-1, 0, 1)$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U = \left(\begin{array}{r|r|r} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix U^{-1}.

The inverse matrix can be computed using row reduction.

Change of basis for a linear operator

Let $L : V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$\begin{array}{ccc} \boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\ U \downarrow & & \downarrow U \\ \boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})} \end{array}$$

It follows that $UA = BU$.

Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$,

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3,1)$, $\mathbf{v}_2 = (2,1)$.

Let S be the matrix of L with respect to the standard basis, N be the matrix of L w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \quad U = \begin{pmatrix}3 & 2\\1 & 1\end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix}1 & -2\\-1 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 2\\1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}\begin{pmatrix}3 & 2\\1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\-1 & 0\end{pmatrix}.$$

Problem. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Find A^{16}.

We already know that vectors $\mathbf{u}_1 = (-1, 1, 0)$, $\mathbf{u}_2 = (1, 1, 0)$, and $\mathbf{u}_3 = (-1, 0, 1)$ are eigenvectors of the matrix A: $A\mathbf{u}_1 = \mathbf{0}$, $A\mathbf{u}_2 = 2\mathbf{u}_2$, $A\mathbf{u}_3 = 2\mathbf{u}_3$. It follows that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, B is the matrix of the operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = A\mathbf{x}$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ while U is the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to the standard basis.

The equality $A = UBU^{-1}$ implies that
$A^2 = AA = UBU^{-1}UBU^{-1} = UB^2U^{-1}$,
$A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$, and so on.
Thus $A^n = UB^nU^{-1}$ for $n = 1, 2, 3, \ldots$
In particular, $A^{16} = UB^{16}U^{-1}$.

$$B^{16} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{16} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^{16} & 0 \\ 0 & 0 & 2^{16} \end{pmatrix} = 2^{15}B.$$

Hence $A^{16} = U(2^{15}B)U^{-1} = 2^{15}UBU^{-1} = 2^{15}A$.

$$A^{16} = 32768\,A = \begin{pmatrix} 32768 & 32768 & -32768 \\ 32768 & 32768 & 32768 \\ 0 & 0 & 65536 \end{pmatrix}.$$