MATH 311-504

Topics in Applied Mathematics

Lecture 2-13:
Review for Test 2.

Topics for Test 2

Vector spaces and linear transformations (Williamson/Trotter 3.1–3.4)

- Vector spaces. Subspaces.
- Linear mappings. Matrix transformations.
- Span. Image and null-space.
- Linear independence (especially in functional spaces).

Basis, dimension, coordinates (Williamson/Trotter 3.5, 3.6C)

- Basis of a vector space. Dimension.
- Matrix of a linear transformation.
- Change of coordinates.

Eigenvalues and eigenvectors (Williamson/Trotter 3.6)

- Eigenvalues, eigenvectors, eigenspaces.
- Characteristic equation of a matrix.
- Bases of eigenvectors, diagonalization.

Sample problems for Test 2

Problem 1 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Sample problems for Test 2

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Sample problems for Test 2

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the image of f.
(ii) Find a basis for the null-space of f.

Sample problems for Test 2

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) For each eigenvalue of B, find an associated eigenvector.
(iii) Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of B? Explain.
(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.
(v) Find all eigenvalues of the matrix B^2.

Sample problems for Test 2

Bonus Problem 5 (20 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, \\ \dfrac{dy}{dt} = x + y + z, \\ \dfrac{dz}{dt} = 2y + z. \end{cases}$$

Problem 1. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0, 0, 0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3 xyz = 0.$

That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1$.

Hence S_1 is closed under scalar multiplication.

However S_1 is not closed under addition.

Counterexample: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

Problem 1. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence S_2 is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0.$
That is, $\mathbf{v} = (x, y, z)$, $\mathbf{v}' = (x, y, z) \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence S_2 is closed under addition.

(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0$.

S_3 is a nonempty set closed under addition and scalar multiplication.

(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

S_4 is a nonempty set closed under scalar multiplication.
However S_4 is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Problem 2. Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix.

By definition, M_L is a 4×4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis E_1, E_2, E_3, E_4.

$$L(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$

$$L(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$

$$L(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Problem 3. Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(\mathbf{x}) = A\mathbf{x}, \text{ where } A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the image of f.

The image of f is spanned by columns of the matrix A:
$\mathbf{v}_1 = (1, -2, -1)$, $\mathbf{v}_2 = (-1, 1, 0)$, $\mathbf{v}_3 = (-2, 3, 1)$.

$$\det A = \begin{vmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -1 & -2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = 0.$$

Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.
It is easy to observe that $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$.
It follows that $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_3)$.

Since the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly independent, they form a basis for the image of f.

Problem 3. Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(\mathbf{x}) = A\mathbf{x}, \text{ where } A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

(ii) Find a basis for the null-space of f.

The null-space of f is the set of solutions of the vector equation $A\mathbf{x} = \mathbf{0}$. To solve the equation, we convert the matrix A to reduced row echelon form:

$$\begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

General solution: $(x, y, z) = (t, -t, t) = t(1, -1, 1),\ t \in \mathbb{R}$.
Hence the null-space is a line and $(1, -1, 1)$ is its basis.

Problem 4. Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix B has three eigenvalues: -1, 1, and 3.

Problem 4. Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(ii) For each eigenvalue of B, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix B associated with an eigenvalue λ is a nonzero solution of the vector equation

$$(B-\lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $B - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$B + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of B associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$B - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of B associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$B-3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(B-3I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

Problem 4. Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iii) Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of B? Explain.

The vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Alternatively, the existence of a basis for $\mathbb{R}^3$ consisting of eigenvectors of B already follows from the fact that the matrix B has three distinct eigenvalues.

Problem 4. Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

Basis of eigenvectors: $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, $\mathbf{v}_3 = (1, 1, 1)$. We have that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Here D is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ while U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis.

Problem 4. Let $B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(v) Find all eigenvalues of the matrix B^2.

Suppose that $B\mathbf{v} = \lambda\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$B^2\mathbf{v} = B(B\mathbf{v}) = B(\lambda\mathbf{v}) = \lambda(B\mathbf{v}) = \lambda^2\mathbf{v}.$$

It follows that $(-1)^2 = 1^2 = 1$ and $3^2 = 9$ are eigenvalues of the matrix B^2. These are the only eigenvalues of B^2.

Indeed, assume that $B^2\mathbf{v} = \mu\mathbf{v}$, where $\mathbf{v} \neq \mathbf{0}$. We have $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3$ for some $r_1, r_2, r_3 \in \mathbb{R}^3$. Then
$B^2\mathbf{v} = r_1(B^2\mathbf{v}_1) + r_2(B^2\mathbf{v}_2) + r_3(B^2\mathbf{v}_3) = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + 9r_3\mathbf{v}_3$,
$\mu\mathbf{v} = \mu r_1\mathbf{v}_1 + \mu r_2\mathbf{v}_2 + \mu r_3\mathbf{v}_3$.

$$\Longrightarrow\ r_1 = \mu r_1,\ r_2 = \mu r_2,\ 9r_3 = \mu r_3$$
$$\Longrightarrow\ (\mu - 1)r_1 = (\mu - 1)r_2 = (\mu - 9)r_3 = 0$$
$$\Longrightarrow\ \mu = 1 \text{ or } \mu = 9$$

Bonus Problem 5. Solve the following system of differential equations (find all solutions):

$$\begin{cases} \frac{dx}{dt} = x + 2y, \\ \frac{dy}{dt} = x + y + z, \\ \frac{dz}{dt} = 2y + z. \end{cases}$$

Let $\mathbf{v} = (x, y, z)$. Then the system can be rewritten in vector form

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v}, \text{ where } B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

Matrix B admits a basis of eigenvectors:
$\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (-1, 0, 1)$, $\mathbf{v}_3 = (1, 1, 1)$.
We have $B\mathbf{v}_1 = -\mathbf{v}_1$, $B\mathbf{v}_2 = \mathbf{v}_2$, $B\mathbf{v}_3 = 3\mathbf{v}_3$.

The vector-function $\mathbf{v}(t)$ is uniquely represented as $\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2 + r_3(t)\mathbf{v}_3$, where $r_1(t)$, $r_2(t)$, and $r_3(t)$ are scalar functions.

$$\frac{d\mathbf{v}}{dt} = \frac{dr_1}{dt}\mathbf{v}_1 + \frac{dr_2}{dt}\mathbf{v}_2 + \frac{dr_3}{dt}\mathbf{v}_3,$$

$$B\mathbf{v} = r_1 B\mathbf{v}_1 + r_2 B\mathbf{v}_2 + r_3 B\mathbf{v}_3 = -r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + 3r_3\mathbf{v}_3.$$

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v} \iff \begin{cases} \frac{dr_1}{dt} = -r_1, \\ \frac{dr_2}{dt} = r_2, \\ \frac{dr_3}{dt} = 3r_3. \end{cases}$$

The general solution: $r_1(t) = c_1 e^{-t}$, $r_2(t) = c_2 e^{t}$, $r_3(t) = c_3 e^{3t}$, where c_1, c_2, c_3 are arbitrary constants.

Thus $\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2 + r_3(t)\mathbf{v}_3 =$
$= c_1 e^{-t}(1, -1, 1) + c_2 e^{t}(-1, 0, 1) + c_3 e^{3t}(1, 1, 1).$

$$\begin{cases} x(t) = c_1 e^{-t} - c_2 e^{t} + c_3 e^{3t}, \\ y(t) = -c_1 e^{-t} + c_3 e^{3t}, \\ z(t) = c_1 e^{-t} + c_2 e^{t} + c_3 e^{3t}. \end{cases}$$