MATH 311-504

Topics in Applied Mathematics

Lecture 2-3:
Subspaces of vector spaces.
Span.

Vector space

A *vector space* is a set V equipped with two operations, **addition**

$$V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$$

and **scalar multiplication**

$$\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V,$$

that have the following properties:

A1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

A2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

A3. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

A4. $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$

A5. $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

A6. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

A7. $(rs)\mathbf{a} = r(s\mathbf{a})$

A8. $1\mathbf{a} = \mathbf{a}$

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- $\mathcal{P}_n$: polynomials of degree at most n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \quad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.

Proof: "only if": the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ is a solution only if all equations are homogeneous.

"if": a system of homogeneous linear equations is equivalent to a matrix equation $A\mathbf{x} = \mathbf{0}$.

$A\mathbf{0} = \mathbf{0} \implies \mathbf{0}$ is a solution $\implies$ solution set is not empty.
If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$.
If $A\mathbf{x} = \mathbf{0}$ then $A(r\mathbf{x}) = r(A\mathbf{x}) = \mathbf{0}$.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$.
Consider the set L of all linear combinations
$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) + (s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n) =$$
$$= (r_1+s_1)\mathbf{v}_1 + (r_2+s_2)\mathbf{v}_2 + \cdots + (r_n+s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1 + (tr_2)\mathbf{v}_2 + \cdots + (tr_n)\mathbf{v}_n.$$

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1,1,0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1,1,1) + t(1,-1,0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- The plane $t_1(1,0,0) + t_2(0,1,1)$, $t_1, t_2 \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$.
- In general, a line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

Examples of subspaces of $\mathcal{M}_2(\mathbb{R})$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric matrices ($A^T = -A$):

$a = d = 0,\ c = -b$

- matrices with zero trace: $a + d = 0$

(trace = the sum of diagonal entries)

- matrices with zero determinant, $ad - bc = 0$,

do not form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Span: implicit definition

Let S be a subset of a vector space V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace of V that contains S. That is,

- $\mathrm{Span}(S)$ is a subspace of V;
- for any subspace $W \subset V$ one has
$$S \subset W \implies \mathrm{Span}(S) \subset W.$$

Remark. The span of any set $S \subset V$ is well defined (it is the intersection of all subspaces of V that contain S).

Span: effective description

Let S be a subset of a vector space V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

- If S is an infinite set then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ ($k \geq 1$).

- If S is the empty set then $\mathrm{Span}(S) = \{\mathbf{0}\}$.

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

Examples.

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$ as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for $\mathcal{M}_{2,2}(\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{w} = (4, -7, 3)$. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

We have to check if there exist $r_1, r_2 \in \mathbb{R}$ such that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$. This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Notice that $\mathbb{R}^2$ is spanned by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ since $(x,y) = x\mathbf{e}_1 + y\mathbf{e}_2$.

Hence it is enough to check that vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ belong to $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\text{Span}(\mathbf{v}_1, \mathbf{v}_2) \supset \text{Span}(\mathbf{e}_1, \mathbf{e}_2) = \mathbb{R}^2.$$

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.