MATH 311-504

Topics in Applied Mathematics

Lecture 2-4:
Span (continued).
Image and null-space.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- $\mathcal{P}_n$: polynomials of degree at most n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$
$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Remarks. The zero vector in a subspace is the same as the zero vector in V. Also, the subtraction in a subspace is the same as in V.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all equations in the system are homogeneous (all $b_i = 0$).

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Definition. The subspace L is called the **span** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and denoted

$$\boxed{\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n).}$$

If $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = V$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is called a **spanning set** for V.

Remark. $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is the minimal subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Examples. • $t\mathbf{x}$, a line through the origin in $\mathbb{R}^n$, is the span of one vector $\mathbf{x} \neq \mathbf{0}$.

• $t\mathbf{x} + s\mathbf{y}$, a plane through the origin in $\mathbb{R}^n$, is the span of two linearly independent vectors $\mathbf{x}$ and $\mathbf{y}$.

$\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

• The span of $\{1, x, x^2\}$ is the space $\mathcal{P}_2$ of polynomials of degree at most 2.

• The span of $\{1, x-1, (x-1)^2\}$ is again $\mathcal{P}_2$.

• The span of $\{1, x, x^2, \dots\}$ is the whole space $\mathcal{P}$.

• The span of $\{x, x^2, x^3, \dots\}$ is the subspace of polynomials $p(x)$ with a root at zero: $p(0) = 0$.

• The span of $\{1, x^2, x^4, \dots\}$ is the subspace of *even* polynomials: $p(-x) = p(x)$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric matrices ($A^T = -A$):

$a = d = 0$ and $c = -b$

- matrices with zero trace: $a + d = 0$

(trace = the sum of diagonal entries)

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ consists of all matrices of the form

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the subspace of anti-symmetric matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the subspace of upper triangular matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the entire space $\mathcal{M}_{2,2}(\mathbb{R})$.

Image and null-space

Let V_1, V_2 be vector spaces and $f : V_1 \to V_2$ be a linear mapping.

V_1: the **domain** of f
V_2: the **range** of f

Definition. The **image** of f (denoted $\mathrm{Im}\, f$) is the set of all vectors $\mathbf{y} \in V_2$ such that $\mathbf{y} = f(\mathbf{x})$ for some $\mathbf{x} \in V_1$. The **null-space** of f (denoted $\mathrm{Null}\, f$) is the set of all vectors $\mathbf{x} \in V_1$ such that $f(\mathbf{x}) = \mathbf{0}$.

Theorem The image of f is a subspace of the range. The null-space of f is a subspace of the domain.

$f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\mathbf{x}) = A\mathbf{x}$, A an m-by-n matrix.

Theorem $\operatorname{Im} f$ is spanned by columns of A.

Proof: Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n,$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis.

$$\implies f(\mathbf{x}) = x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n).$$

Hence the image of f is spanned by vectors $f(\mathbf{e}_1), f(\mathbf{e}_2), \dots, f(\mathbf{e}_n)$, which are columns of A.

The null-space of f is the solution set of a system of linear equations, $A\mathbf{x} = \mathbf{0}$.

Proposition $\operatorname{Null} f$ is not changed when we apply elementary *row* operations to the matrix A.

Examples

- $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\mathrm{Im}\, f$ is spanned by vectors $(1, 1, 1)$, $(0, 2, 0)$, and $(-1, -1, -1)$. It follows that $\mathrm{Im}\, f$ is the plane $t(1, 1, 1) + s(0, 1, 0)$.

To find $\mathrm{Null}\, f$, we convert A to reduced form:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $(x, y, z) \in \mathrm{Null}\, f$ if $x - z = y = 0$.
It follows that $\mathrm{Null}\, f$ is the line $t(1, 0, 1)$.

- $f : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}),\ \ f(A) = A + A^T.$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}.$$

$\mathrm{Null}\, f$ is the subspace of anti-symmetric matrices, $\mathrm{Im}\, f$ is the subspace of symmetric matrices.

- $g : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R}),\ \ g(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A.$

$$g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

$\mathrm{Im}\, g$ is the subspace of matrices with the zero second row, $\mathrm{Null}\, g$ is the same as the image $\implies\ g(g(A)) = O$.

$\mathcal{P}$: the space of polynomials.
$\mathcal{P}_n$: the space of polynomials of degree at most n.

- $D : \mathcal{P} \to \mathcal{P}$, $(Dp)(x) = p'(x)$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$
$$\implies (Dp)(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$$

The image of D is the entire $\mathcal{P}$, $\mathrm{Null}\, D = \mathcal{P}_0 =$ the subspace of constants.

- $D : \mathcal{P}_3 \to \mathcal{P}_3$, $(Dp)(x) = p'(x)$.

$$p(x) = ax^3 + bx^2 + cx + d \implies (Dp)(x) = 3ax^2 + 2bx + c$$

The image of D is $\mathcal{P}_2$, $\mathrm{Null}\, D = \mathcal{P}_0$.